Theory 6: 'Bolometer'

The entrance cavity of a particular bolometer is a cone with an opening angle of 30°, the surface of which has an energy absorption coefficient of $a = 0.99$. Assume that there is no scattering of the incident radiation on the walls of the cavity, only multiple specular reflections. The bolometer is connected to a cooler which keeps the bolometer cavity surface at practically 0 K temperature. The instrument is orbiting at 2 au from the Sun and is pointed directly at the centre of the Solar disk.

Calculate the temperature of a black body which would radiate the same amount of energy from a unit surface area as the bolometer opening does.

Note: the opening angle is defined as twice the angle between the axis of the cone and its generatrix.

(13 points)

Solution

From the table of constants, the Solar luminosity $L_\odot = 3.826 \times 10^{26}$ W and $1\,\mathrm{au} = 1.496 \times 10^{11}$ m, therefore the distance to the bolometer is:

$$r = 2\,\mathrm{au} = 2.992 \times 10^{11}\,\mathrm{m}$$

and the surface area A of a sphere of that radius is:

$$A_\mathrm{sphere} = 4\pi r^2 = 1.125 \times 10^{24}\,\mathrm{m}^2$$

and the incident flux is:

$$F(r) = L_\odot/A_\mathrm{sphere} = 340.1\,\mathrm{W\,m}^{-2} \qquad \text{(3 points)}$$

The incoming radiation can be assumed to be initially parallel to the axis of the cone. As the rays hit the surface they are reflected, after the first reflection the rays are travelling at $30°$ to their original path ($2 \times 15°$). They next meet the surface at $30° + 15° = 45°$ and are reflected by $90°$, and so on with each reflection, such that after $N = 6$ reflections the ray will be sent back out of the aperture as shown in the figure below. These exiting rays are what we are interested in; all the energy entering the bolometer itself is removed by the 100% efficient cooler.
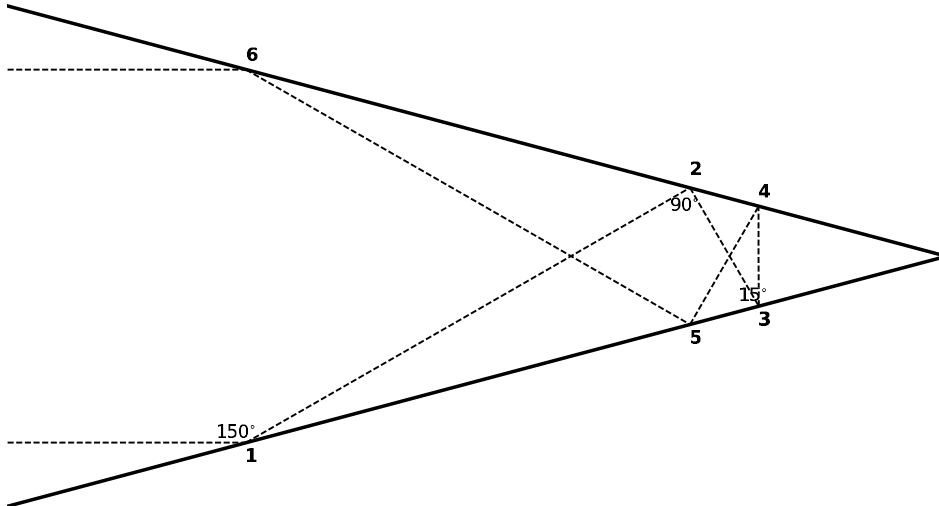
(conceptually difficult: 4 points)

Thus the fraction of energy leaving the bolometer will be given by:

$$S = (1-a)^N = 0.01^6 = 10^{-12}. \qquad \text{(2 points)}$$

Treating the opening as if it were a black body radiator, we can apply the Stefan-Boltzmann law:

$$\Phi = \sigma T^4 \qquad \text{(1 point)}$$

$$\implies T = (\Phi/\sigma)^{0.25},$$

and

$$\Phi = F(r) \cdot S = 340.1 \times 10^{-12}\,\mathrm{Wm}^{-2} \qquad \text{(2 points)}$$

From the table of constants: $\sigma = 5.670 \times 10^{-8}\,\mathrm{Wm}^{-2}\mathrm{K}^{-4}$, therefore the effective temperature is:

$$T = \left(\frac{340.1 \times 10^{-12}}{5.670 \times 10^{-8}}\right)^{0.25} = 0.28\,\mathrm{K} \qquad \text{(1 point)}$$

Theory 7: 'Libration'

As a result of libration, studied among others by Johannes Hevelius, more than half of the Moon's surface can be observed from Earth. Assume that the observer is geocentric.
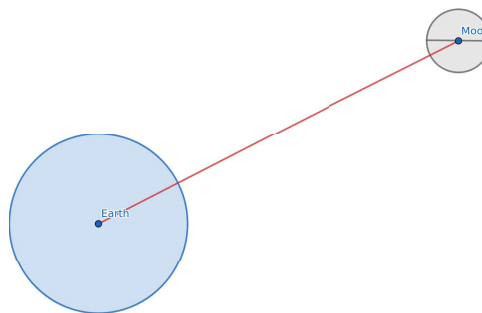
(a) Estimate ϕ_B, the maximum angle of libration in latitude. The axial tilt (obliquity) of the Moon with respect to its orbital plane is $\alpha = 6°41'$.

(b) Estimate ϕ_L, the maximum angle of libration in longitude. Assume that the Moon is always aligned with the same side facing towards the second focus F2 of its orbit, and that the eccentricity of the Moon's orbit e changes between 0.044 and 0.064 on a timescale of several months.

(c) Estimate the fraction of the Moon's surface which can be seen from Earth.

(d) Calculate how many months (lunations) are needed for an observer to see the Moon's surface determined in part (c).
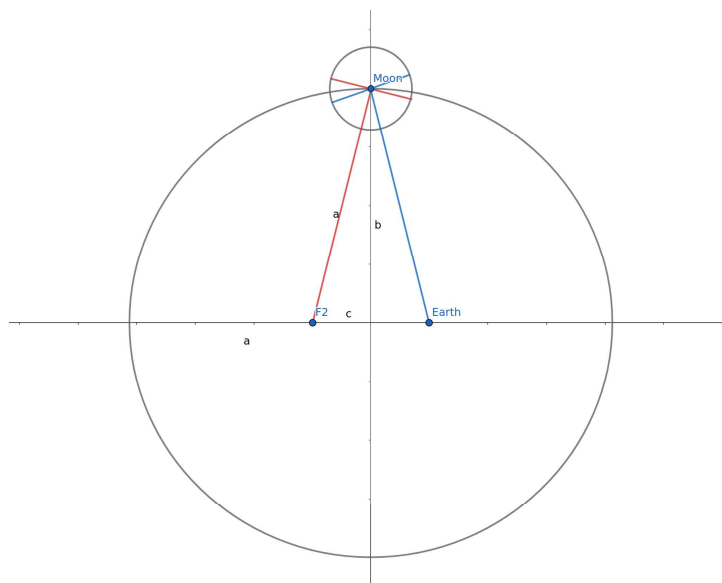
(20 points)

Solution

(a) $\phi_B = \alpha = 6°41'$ (*conceptually difficult:* 5 points)

(b) The maximum angle of libration ϕ_L occurs when the Moon is at point M and for $e = 0.064$:

ϕ_L is the Earth–Moon–F2 angle; $|EF2| = 2c$.

Therefore $\sin(\phi_L/2) = c/a = e \implies \phi_L = 2\arcsin(0.064) = 7.34° = 7°20'$ (5 points)

(c) In the less accurate approximation,

$$(2\phi_B + 2\phi_L)/180° = x/50\% \implies x = 8\%$$

and the total area visible is $S \approx 50\% + x = 58\%$.

This is less accurate as it counts the overlapping areas made visible by both librations twice.

A slightly more accurate approximation is given by taking the average of ϕ_B and ϕ_L, $7°01'$. The belt made visible by libration will thus be $2\pi R_{\text{Moon}} \cdot 7°/360° \approx 213$ km wide. The surface area of the belt is then approximately $213 \text{ km} \times 2\pi R_{\text{Moon}} = 2.32 \times 10^6 \text{ km}^2$, which accounts for about 6% of the lunar surface. If the students does this algebraically before substituting, they also get:

$$\frac{2\pi R_{\text{Moon}} \cdot 2\pi R_{\text{Moon}}}{4\pi 2 R_{\text{Moon}}^2} \cdot \frac{7°}{360°} = \pi \frac{7°}{360°} \approx 6\%.$$

In either case the total area visible is $S \approx 50\% + 6\% = 56\%$.

(less accurate solution 4 points, better solution 5 points)

(d) It is necessary to find a common multiple of the period of libration in latitude $P_B = 27.2122$ d (the draconic month) and the period of libration in longitude $P_L = 27.5545$ d (the anomalistic month).

The simplest way is analogous to synodic periods:

$$\frac{1}{T_{\text{syn}}} = \frac{1}{T_1} - \frac{1}{T_2}$$

thus

$$\frac{1}{T} = \frac{1}{P_B} - \frac{1}{P_L} \implies T \approx 2190.5 \,\text{d} = 74 \,\text{lunations}.$$

Or using continued fractions:

$$\frac{P_L}{P_B} = \frac{275545}{272122} \approx 1 + \cfrac{1}{79 + \cfrac{1}{2 + \cfrac{1}{131 + \cfrac{1}{6 + \dots}}}}.$$

With the first term we approximate $P_L/P_B \approx 80/79$; since $80 \times 27.2122 \approx 2176.9$ d and $79 \times 27.5545 \approx 2176.8$ d this is already close enough, and T is again equal to about 74 lunations.

(3 points)

Since we only need half the cycle to see everything once (as the Moon swings away and back again during the whole cycle) therefore $T/2 = 37$ months ≈ 3 years is enough to see all of the potentially visible parts of the Moon's surface.

(2 points)

Theory 8: 'Neutrinos'

In a simplified model of a supernova explosion, the core of a star, composed of pure iron $^{56}_{26}\text{Fe}$ nuclei with a total mass of 1 $M_\odot$, changes into a neutron star composed of individual electrons, protons and neutrons in numerical proportions of 1:1:8. This process is called 'neutronization' and results in the emission of a large number of neutrinos.

Calculate the solar neutrino flux on Earth. How much larger would the flux of neutrinos reaching the Earth from the supernova be than the steady neutrino emission of the Sun, if the supernova exploded in the centre of the Galaxy and the process of neutronization of the core took about 0.01 s? Give an order-of-magnitude answer.

(20 points)

Solution

Neutrino emission of the supernova:

From the table of constants, the core mass $M = 1 M_\odot = 1.988 \cdot 10^{30}$ kg.
Mass of one atom of iron $^{56}_{26}\text{Fe}$:

$$m_{\text{Fe}} = 56 \,\text{Da} = 56 \times 1.661 \times 10^{-27} \,\text{kg} = 9.3016 \times 10^{-26} \,\text{kg}.$$

> Note: atomic mass is expressed in *atomic mass units* symbol a.m.u. or u, also called *Daltons*, symbol Da. Students may use any of these notations in their solutions.

The number of $^{56}_{26}\text{Fe}$ nuclei in the core is therefore:

$$n_{\text{Fe}} = M/m_{\text{Fe}} = 2.1373 \times 10^{55} \qquad \text{(1 points)}$$

and the initial number of nucleons is:

$$n_{\text{nuc}} = 56 \, n_{\text{Fe}} = 1.1969 \times 10^{57}. \qquad \text{(1 points)}$$

Correspondingly, the initial number of protons is $n_{\text{p}} = 26 \, n_{\text{Fe}}$ and the initial number of neutrons is $n_{\text{n}} = 30 \, n_{\text{Fe}}$. (2 points)

We assume all the nucleons in the original $^{56}_{26}\text{Fe}$ nuclei are converted to individual nucleons in the given proportions (1 proton : 8 neutrons). Therefore after the explosion,

$$n'_{\text{n}} = \frac{8}{9} n_{\text{nuc}} = 1.0634 \times 10^{57}$$

neutrons and

$$n'_{\text{p}} = \frac{1}{9} n_{\text{nuc}} = 1.3299 \times 10^{56}$$

protons remain, and thus

$$n_{\text{p}} - n'_{\text{p}} = 26 \, n_{\text{Fe}} - n'_{\text{p}} = 4.2271 \times 10^{56} \qquad \text{(4 points)}$$

protons are changed. Since 1 neutrino is produced for every converted proton ($\text{p} + \text{e}^- \rightarrow \text{n} + \nu_{\text{e}}$), the neutrino flux is:

$$I_\nu = \frac{4.2271 \times 10^{56}}{0.01 \,\text{s}} = 4.2271 \times 10^{58} \,\text{neutrinos s}^{-1}. \qquad \text{(2 points)}$$
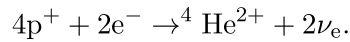
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The flux density observed at Earth is given by:

$$F_{\text{SN}} = \frac{I_\nu}{4\pi d_{\text{Gal}}^2}.$$

Substituting $d_{\text{Gal}} = 8 \text{ kpc} = 8000 \times 3.086 \times 10^{16} \text{ m} = 2.4688 \times 10^{20} \text{ m}$,

$$F_{\text{SN}} \approx 5.5 \times 10^{16} \text{ neutrinos s}^{-1} \text{ m}^{-2}. \hspace{2cm} \text{(2 points)}$$

Neutrino flux from the Sun:

We can assume that the primary source of Solar luminosity is the p–p reaction,

$$4\text{p}^+ + 2\text{e}^- \rightarrow {}^4\text{He}^{2+} + 2\nu_{\text{e}}.$$

Neglecting the electrons and neutrinos, the change in mass Δm is:

$$\Delta m = 4 \cdot 1.673 \times 10^{-27} \text{ kg} - 6.645 \times 10^{-27} \text{ kg} = 4.7 \times 10^{-29} \text{ kg},$$

and thus from $E = (\Delta m)c^2$ the amount of energy released is $\approx 4.2244 \times 10^{-12} \text{ J}$.
Solar luminosity is $L_\odot = 3.826 \times 10^{26} \text{ W}$, so the number of reactions per second is approximately:

$$\frac{3.826 \times 10^{26}}{4.2244 \times 10^{-12}} = 9.057 \times 10^{37}.$$

With 2 neutrinos released per reaction, this gives a Solar neutrino flux of:

$$I_{\nu\odot} \approx 1.8 \times 10^{38} \text{ neutrinos s}^{-1}.$$

(If the student remembers that the p–p reaction releases 26.73 MeV $= 4.2826 \times 10^{-12}$ J, they will get 8.9338×10^{37} reactions per second and the same final answer.)

$$\text{(getting to } I_{\nu\odot}\text{: 5 points)}$$

At the distance of Earth, the observed flux density is given by:

$$F_\odot = \frac{I_{\nu\odot}}{4\pi d_\odot^2},$$

where $d_\odot = 1 \text{ au} = 1.496 \times 10^{11} \text{ m}$, and so:

$$F_\odot \approx 6.4 \times 10^{14} \text{ neutrinos s}^{-1} \text{ m}^{-2}. \hspace{2cm} \text{(1 point)}$$

The final ratio is therefore:

$$F_{SN}/F_\odot \approx \frac{5.5 \times 10^{16}}{6.4 \times 10^{14}} \approx 86 \approx 100,$$

or two orders of magnitude. $\hspace{6cm} \text{(2 points)}$

Theory 9: 'Second eclipse'

For each of two eclipsing binary systems, Bolek and Lolek, the primary eclipses were observed with very high cadence as depicted below:
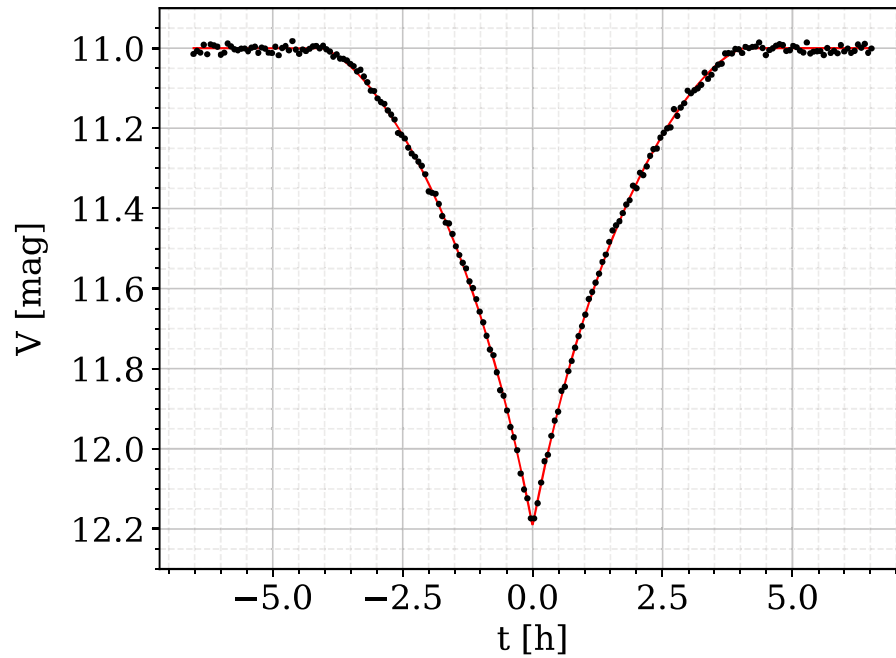

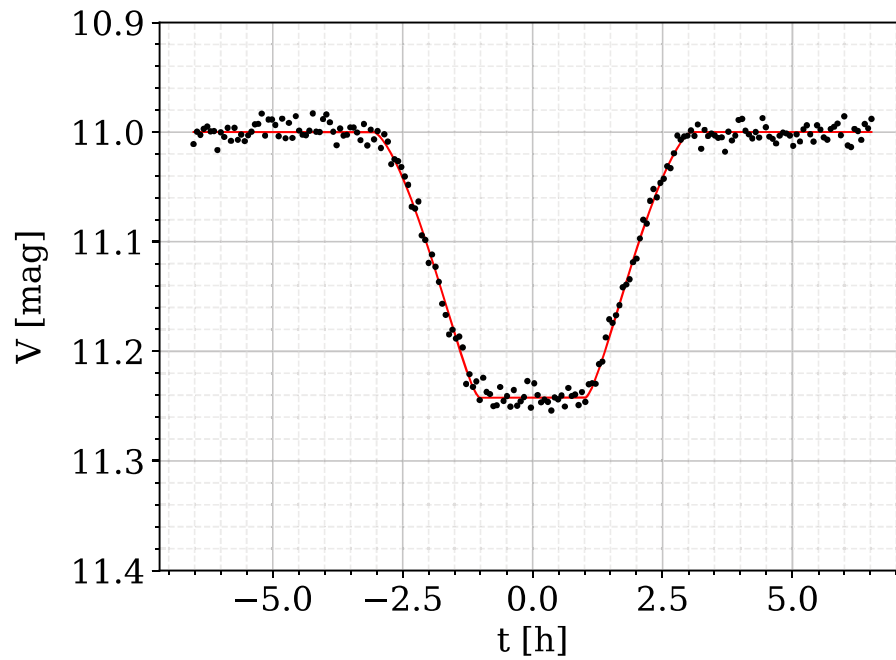Figure 1: Observed lightcurve for system Bolek.

Figure 2: Observed lightcurve for system Lolek.

In the figures, t is the time in hours relative to the moment of minimum and V is the brightness in the V (visible) band in magnitudes. The points are the measurements and the line is the fitted model of the shape of the eclipse.

You can assume that in both cases the eclipses are central ($i = 90°$) and last for a very small fraction of the orbital period, limb darkening is negligible, and the orbits have low eccentricity.

On the Answer Sheet, draw the predicted shape of the light curve for each of the secondary eclipses. Write down the equations and calculations leading to your predictions.
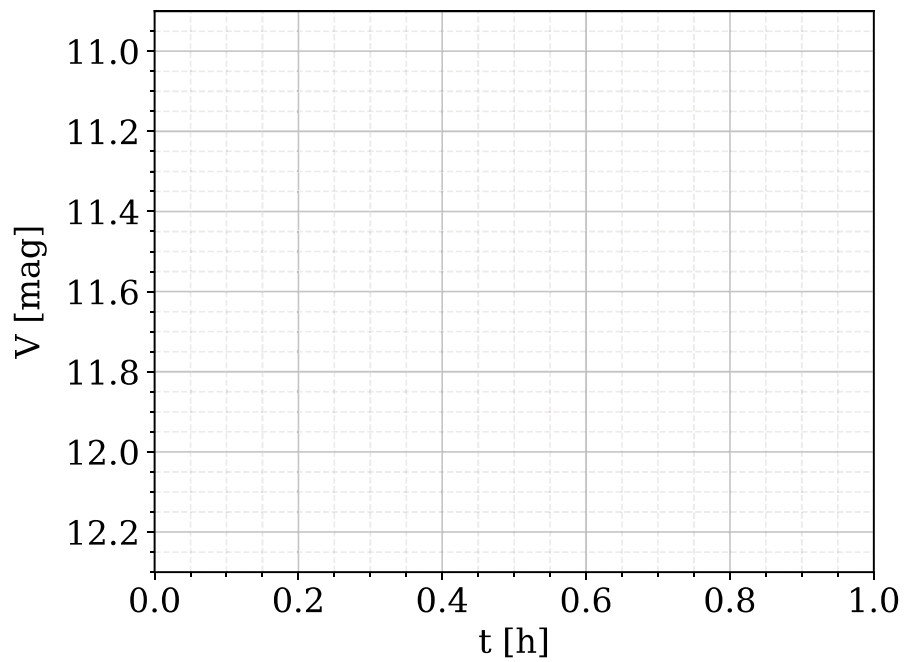
(20 points)

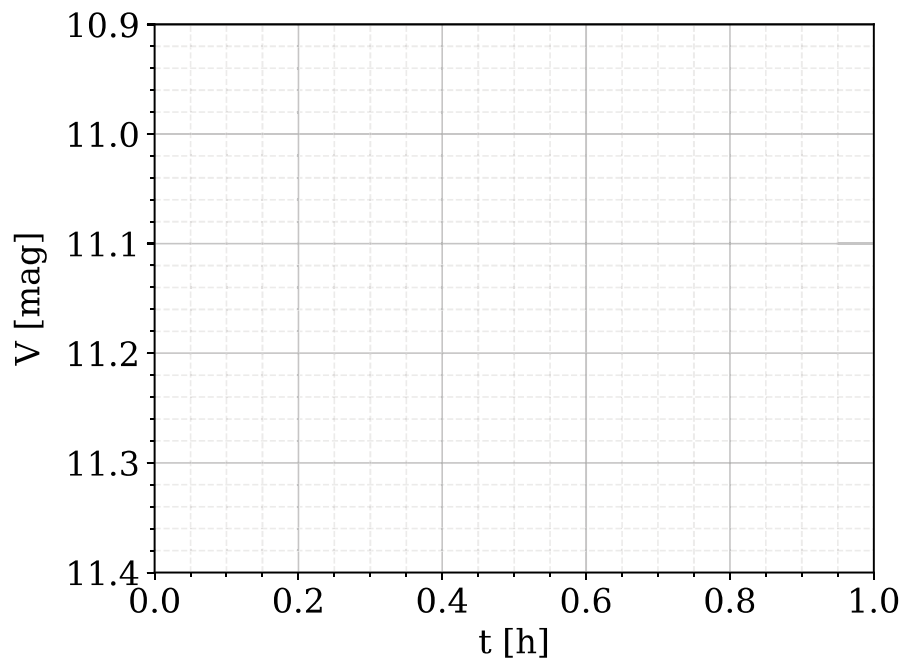Figure 3: Predicted lightcurve for the second eclipse for system Bolek.

Figure 4: Predicted lightcurve for the second eclipse for system Lolek.

Solution

For each of the systems, since the eclipses are short we can assume that the angle swept along the orbit during the eclipse is negligibly small, which means that the tangential velocity in relative orbital motion is constant.

System A

A sharp peak minimum in a central eclipse implies that $R_1 = R_2$. Therefore, both eclipses will be total, with minima of brightnesses m_1 and m_2, respectively.

Noticing that $R_1 = R_2$	1 point
Brightnesses at the two minima are m_1 and m_2	1 point

Define the magnitude at minimum as $m_{\min} = m_1$ and the baseline magnitude as $m_{1,2} = m_1 + m_2$. From Pogson's law applied to the two cases: $m_1 - m_{1,2} = -2.5 \log_{10}(L_1/(L_1 + L_2))$ and $m_2 - m_1 = -2.5 \log_{10}(L_2/L_1)$. For the first case, with simple operations we obtain: $L_2/L_1 = 10^{(m_1 - m_{1,2})/2.5} - 1$. Plugging this into the second case: $m_2 = m_1 - 2.5 \log_{10}\left(10^{(m_1 - m_{1,2})/2.5} - 1\right)$.

Correct application of Pogson's law	1 point
Obtaining the luminosity ratio	1 point
Final formula for the depth of the minimum	1 point

As the orbits are near-circular, timescales will be the same for both eclipses. Therefore it is sufficient to 'copy' the shape of the primary one while scaling down the depth to $|m_{1,2} - m_2|$. Reading off $m_{1,2}$ and $m_{\min}$ from the plot, we arrive at $L_2/L_1 = 2$, $m_2 = 11.44$, and the following figure:
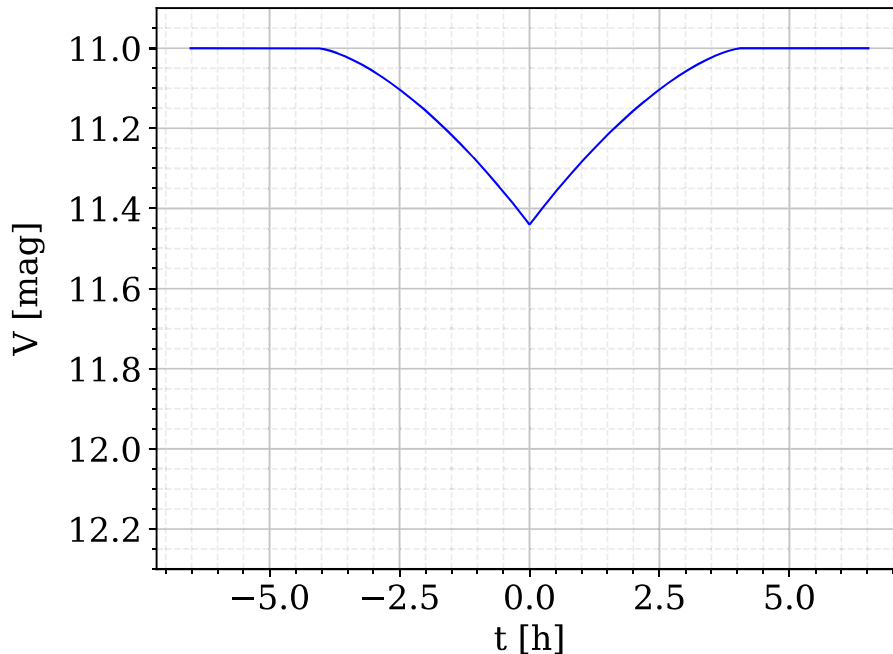

Figure 5: Predicted lightcurve for the second eclipse for system A – correct answer.

Correct depth of the second eclipse	2 points
Overall correct shape (sharp peak and symmetry not broken)	1 point
All contact times matching the first eclipse	1 point

Note about marking: Reproducing the exact shape of the eclipse (concave/convex, smooth descent, etc.) is not subject to grading as it requires mm-level precision, and the focus is on the thought process instead.

System B

For the second system, we follow a generally similar scheme, but we see that the minimum is flat. t_{fall} is the time of the descent from the baseline brightness to the lowest point, i.e., between contacts I and II, while t_{flat} is the time between contacts II and III during which the lightcurve remains at its lowest point:
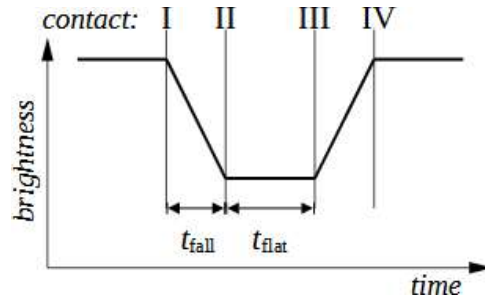
From this, we obtain $t_{\text{flat}}/t_{\text{fall}} = 2(R_2 - R_1)/2R_1$ and, after rearranging, $R_2/R_1 = t_{\text{flat}}/t_{\text{fall}} + 1$. 🗨

Noticing that $R_1 \neq R_2$ and R_2/R_1 can be calculated from the contact times	1 point
Calculating, or giving the correct formula for, R_2/R_1	1 point

Here R_1 is the radius of the smaller star and there are two solutions – it could either be (partially) *eclipsing* or (entirely) *eclipsed*. Let us first assume that it was eclipsed. After measuring m_{min} for the eclipse and $m_{1,2}$ for the baseline we again (just like for system A) assume $m_{\text{min}} = m_1$ and write down: $L_1/L_2 = 10^{(m_1 - m_{1,2})/2.5} - 1$.

Calculating, or giving the correct formula for, L_2/L_1	1 point

Reading off $m_{1,2}$ and m_{min} from the plot, we arrive at $L_2/L_1 = 4$, $R_2/R_1 = 2$.
$L_i = S_i \pi R_i^2$, where S_i stands for the surface brightness of each star.
$\implies S_1 = S_2 = S$ (or $T_{\text{eff},1} = T_{\text{eff},2} = T_{\text{eff}}$).

Relation between L, R ratios and S (or T_{eff}) ratios	1 point
Concluding that $S_1 = S_2$ (or $T_{\text{eff},1} = T_{\text{eff},2}$)	2 points

In such a case, *it does not matter* if star 2 is eclipsing or eclipsed. During the baseline, we always see two disks with a total brightness of $S(\pi R_1^2 + \pi R_2^2)$, while during the eclipse, we see one disk of radius R_2 and total brightness $S\pi R_2^2$.

Note about marking: This solution can be independently verified by assuming that the smaller component was eclipsing and cutting out a disk of surface πR_1^2 and brightness contribution $S_1 \pi R_1^2$ in the star of surface πR_2^2 and surface brightness S_2. Instead of $m_{\text{min}} = m_1$ of

star 1, we would have measured $m_{\min} = m_{ecl,1}$ of a combination of the two, with a brightness $L_{ecl1} = S_1\pi R_1^2 + S_2\pi(R_2^2 - R_1^2)$ The same conclusion $S_1 = S_2 = S$ will follow. Any solution correctly arriving at this conclusion and the correct figure – no matter the order of steps taken – should be scored equally. For the maximum number of points, however, the student should make a convincing point that this is the *only* solution, i.e. lifting the previously made assumption (or exploring both cases).

Solution showing there is only one possible shape of the secondary eclipse	1 point

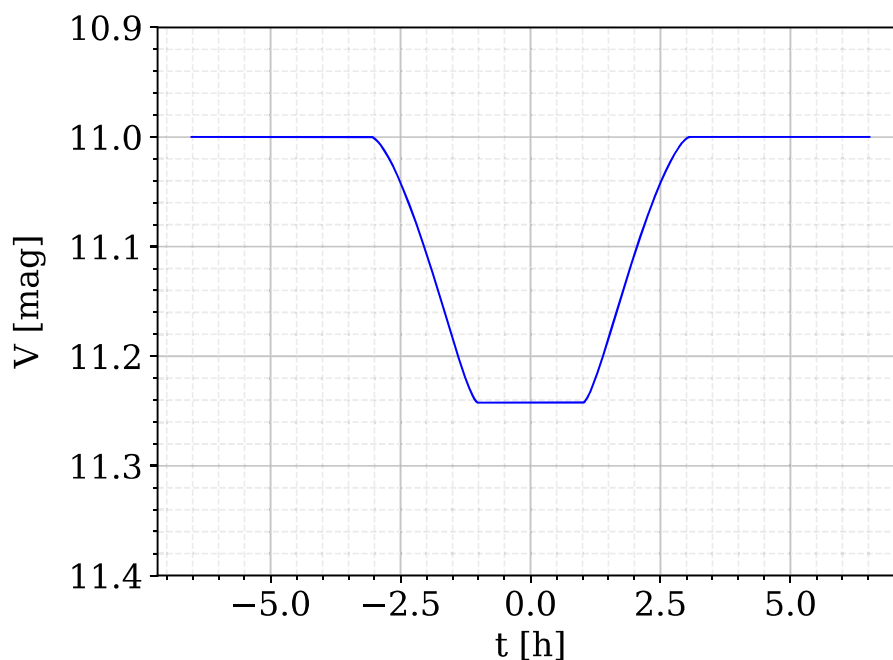To draw the second eclipse, one must simply copy the shape of the first one:

Figure 6: Predicted lightcurve for the second eclipse for system B – correct answer.

Correct depth of the second eclipse	2 points
Overall correct shape (flat minimum and symmetry not broken)	1 point
All contact times matching the first eclipse	1 point

Theory 10: 'Aldebaran'

On 9 March 1497, Nicolaus Copernicus observed the occultation of Aldebaran by the Moon from Bologna. In his work *De revolutionibus orbium cœlestium*[1] Copernicus described the event: "I saw the star touching the dark edge of the Moon and disappearing at the end of the 5th hour of the night between the horns of the Moon, closer to the south horn by a third of the Moon's diameter."

Assuming that the occultation was observed on the local meridian, that at maximum occultation Aldebaran was $0.32'$ above the southern edge of the Moon, and that the apparent angular diameter of the Moon as seen from Bologna was $31.5'$, solve the following tasks:

(a) Find the latitude φ_1 of a place with the same longitude as Bologna, from which Aldebaran would have appeared to pass behind the centre of the Moon.

(b) Find the duration of the occultation as seen from latitude φ_1 if Aldebaran appeared to pass along the diameter of the lunar disk. For simplicity, also assume that the Moon and the observer are moving linearly at constant speed, that the Moon's orbit is circular and that the declination of the Moon does not change during the occultation.

(c) Find the topocentric angular velocity of the Moon against the background stars during the occultation for an observer at latitude φ_1, in arcmin/hour, applying the same assumptions as in part (b).

(d) Estimate the range of the Moon's topocentric angular velocities (against the background stars) in arcmin/hour at latitude φ_1, assuming a circular orbit. Show how this result can be justified by expressing the relative velocity of the Moon and observer in terms of their velocity vectors.

The declination of Aldebaran was $\delta_A = 15.37°$ in 1497 (due to precession), and the latitude of Bologna is $\varphi_B = 44.44°$ N.
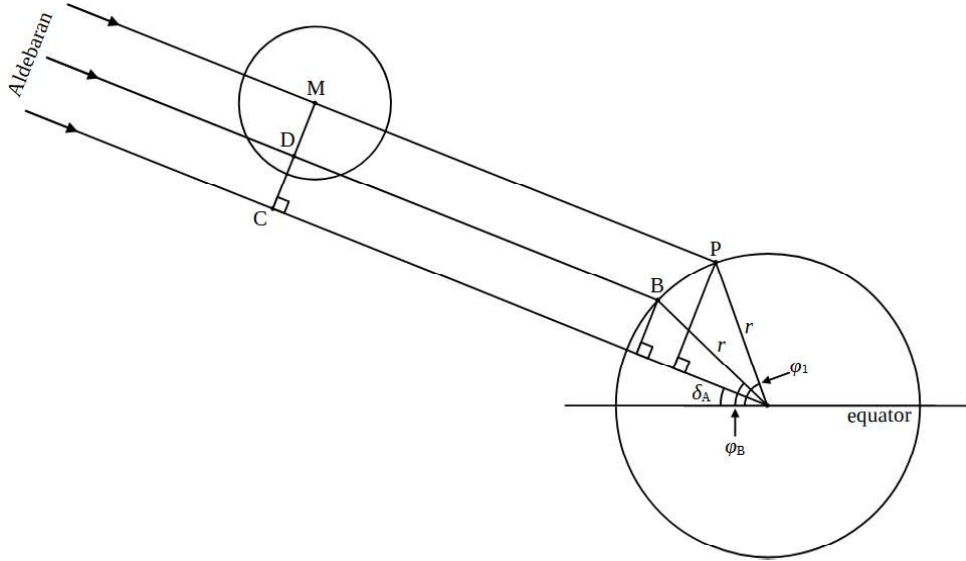
(25 points)

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[1]Book VI, Chapter 27

Solution

(a) Aldebaran is far enough away that the light rays from it can be considered to be parallel, thus the problem can be modelled by the following diagram:

where the light ray through C passes through the centre of the Earth and thus δ_A is the declination of Aldebaran; the ray through D and B (Bologna) represents the situation observed by Copernicus, with Aldebaran behind the Moon and $0.32'$ above the south edge; and the ray through M and P represents the situation where Aldebaran is exactly behind the centre of the Moon. φ_B is the geocentric latitude of Bologna, φ_1 is the latitude of the place P and r is the radius of the Earth (assume the Earth is spherical).

Calculating the latitude φ_1 is thus a matter of simple geometry, where:

$$|CD| = r\sin(\varphi_B - \delta_A),$$

$$|CM| = r\sin(\varphi_1 - \delta_A),$$

and the distance $|DM|$ is the fraction of the Moon's radius given by:

$$|DM| = 1\,737\,\text{km} \times \left(1 - \frac{0.32'}{31.5'/2}\right) = 1\,702\,\text{km}.$$

Therefore, since

$$|CM| = |CD| + |DM| = 3\,099\,\text{km} + 1\,702\,\text{km} = 4\,801\,\text{km},$$

$$\implies \varphi_1 = \arcsin\left(\frac{4\,801}{6\,378}\right) + \delta_A = 64.19°. \tag{6 points}$$

(b) Given the assumptions, the duration of the occultation will just depend on the difference between the tangential speeds of the observer, v_{obs}, and Moon, v_{Moon}, and on the diameter of the Moon.

For the observer, the tangential speed is given by the circumference of their path at latitude φ_1 divided by the sidereal day:

$$v_{\text{obs}} = \frac{2\pi r_{\text{Earth}} \cos\varphi_1}{23^{\text{h}}\,56\text{m}\,4^{\text{s}}} = 729\,\text{km/h}$$

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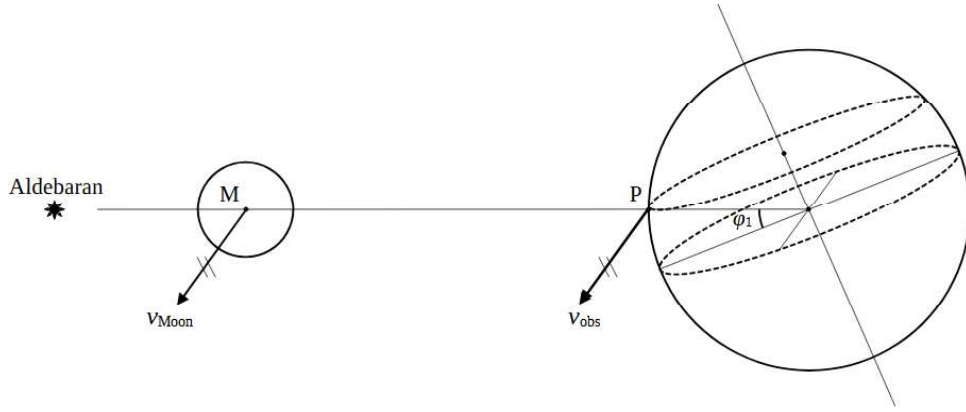
Similarly for the Moon, using the semi-major axis of the Moon's orbit and the length of the sidereal month (from the table of constants):

$$v_{\text{Moon}} = \frac{2\pi \times 3.844 \times 10^5 \,\text{km}}{27.321661 \,\text{d} \times 24} = 3\,683 \,\text{km/h}$$
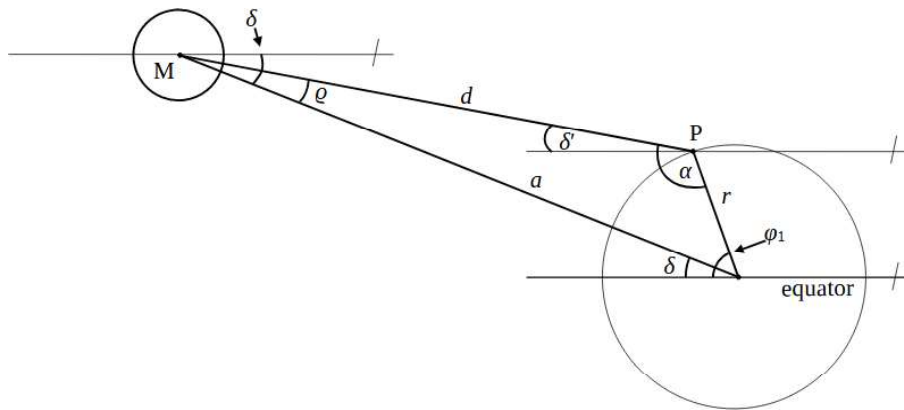
The difference of speeds is then:

$$v_{\text{rel}} = v_{\text{Moon}} - v_{\text{obs}} = 2\,954 \,\text{km/h}$$

and the time taken to move across the lunar diameter ($2 \times 1737 \,\text{km}$) at that speed is:

$$t = \frac{2 \times 1\,737 \,\text{km}}{2\,954 \,\text{km/h}} = 1.176 \approx 1.18 \,\text{hours.} \qquad \text{(5 points)}$$

(c) To determine the topocentric angular velocity, we need to find the distance d between the observer (at place P at latitude φ_1) and the Moon:

where a is the semi-major axis of the Moon's orbit, δ is the geocentric declination of the Moon, and δ' is the topocentric declination of the Moon as seen by the observer. We can rewrite $\varrho = \delta - \delta'$ and $\alpha = 180° - (\varphi_1 - \delta) - \varrho = 180° - (\varphi_1 - \delta) - (\delta - \delta') = 180° - (\varphi_1 - \delta')$, then using the sine theorem:

$$\frac{d}{\sin(\varphi_1 - \delta)} = \frac{a}{\sin\left(180° - (\varphi_1 - \delta')\right)} = \frac{r}{\sin(\delta - \delta')}$$

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Simplifying and substituting $\delta' = \delta_A$:

$$\frac{d}{\sin(\varphi_1 - \delta)} = \frac{a}{\sin(\varphi_1 - \delta_A)} = \frac{r}{\sin(\delta - \delta_A)}$$

We can now calculate δ:

$$\delta = \arcsin\left(\frac{r\sin(\varphi_1 - \delta_A)}{a}\right) + \delta_A = 16.08°$$

and d:

$$d = a\frac{\sin(\varphi_1 - \delta)}{\sin(\varphi_1 - \delta_A)} = 380\,203\,\text{km}.$$

Thus the angular velocity is:

$$\omega = \left(\frac{2\,R_{\text{Moon}}}{d}\right)\left(\frac{1}{1.176\,\text{h}}\right)\text{rad/h} = \left(\frac{2\times 1\,737}{380\,203}\right)\left(\frac{1}{1.176\,\text{h}}\right)\left(\frac{180\times 60}{\pi}\right) = 26.71 \approx 27\,\text{arcmin/h}.$$

(7 points)

(d) The result in part (c), $27\,\text{arcmin/h}$, was calculated for the situation when the Moon and observer are moving in the same direction, reducing their relative velocity, and thus represents the lower limit of the range of topocentric angular velocities. The upper limit is given by the situation when the Moon and observer are moving in the opposite direction, i.e. when the Moon is due north.

Using the calculation from part (b), we find that the relative linear speed is:

$$v'_{\text{rel}} = v_{\text{Moon}} + v_{\text{obs}} = 3\,683 + 729 = 4\,412\,\text{km/h}$$

Neglecting the effect of the slightly greater distance between the observer and the Moon when the Moon is on the northern side of the sky, the angular speed will be greater by the ratio of the linear speeds,

$$\omega' = \omega\,\frac{4\,416}{2\,954} = 39.89 \approx 40\,\text{arcmin/h},$$

thus the range is approximately 27–40 arcmin/h. (3 points)

If the additional distance is taken into account, the upper limit will be around 1% less.

Expressing the velocities as vectors,

$$\vec{v}_{\text{rel}} = \vec{v}_{\text{Moon}} - \vec{v}_{\text{obs}},$$

the magnitude of the relative velocity, $v_{\text{rel}} = |\vec{v}_{\text{rel}}|$ is given by the scalar projection of $\vec{v}_{\text{obs}}$ on $\vec{v}_{\text{Moon}}$:

$$v_{\text{rel}} = v_{\text{Moon}} - v_{\text{obs}}\cos\alpha$$

where α is the angle between the vectors. From this it can be seen that the relative velocity must be minimum for $\alpha = 0°$ and maximum for $\alpha = 180°$. These conditions are met on the local meridian, due south or due north, respectively.

(4 points)