IOAA 2023 General Marking Scheme

Using incorrect physical concept (despite correct answers)	No points given
Giving correct answer without detailed calculation	Deduct 50% of the marks for that part
Minor mistakes in the calculations, e.g., wrong signs, symbols, substitutions	Deduct 20% of the marks for that part
Units missing from final answers	Deduct 0.5 pts
Too few or too many significant figures in the final answer	Deduct 0.5 pts
Error resulting from another error in an earlier part for which the student already lost marks, if the answer is physically reasonable.	Full points (i.e., no deductions)
Error resulting from another error in an earlier part, where the student should have realised the answer was physically unreasonable.	Deduct 20% of the marks for that part

For example, if due to an error in an earlier part, the student calculates the mass of a star as 2.5×10^{30} kg *instead of* 2×10^{30} kg*, they will only lose marks for the earlier part. However, if, for the same reason, a student calculates the mass as* 2×10^{25} kg*, they should realize this is wrong (a few times the Earth's mass) and thus should lose some marks for this part as well.*

Table of Constants

Fundamental constants

Speed of light in vacuum	c	=	$2.998 \times 10^{8}\ \mathrm{m\,s^{-1}}$
Planck constant	h	=	$6.626 \times 10^{-34}\ \mathrm{J\,s}$
Boltzmann constant	k_B	=	$1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	=	$5.670 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$
Elementary charge	e	=	$1.602 \times 10^{-19}\ \mathrm{C}$
Universal gravitational constant	G	=	$6.674 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Universal electric constant	ϵ_0	=	$8.854 \times 10^{-12}\ \mathrm{m^{-3}kg^{-1}s^{4}A^{2}}$
Universal gas constant	R	=	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro constant	N_{A}	=	$6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
Wien's displacement constant	$b = \lambda_m T$	=	$2.898 \times 10^{-3}\ \mathrm{m\,K}$
Mass of electron	m_{e}	=	$9.109 \times 10^{-31}\ \mathrm{kg}$
Mass of proton	m_{p}	=	$1.673 \times 10^{-27}\ \mathrm{kg}$
Mass of neutron	m_{n}	=	$1.675 \times 10^{-27}\ \mathrm{kg}$
Mass of Helium nucleus	m_{He}	=	$6.645 \times 10^{-27}\ \mathrm{kg}$
Atomic mass unit (a.m.u., Dalton)		=	$1.661 \times 10^{-27}\ \mathrm{kg}$

Astronomical data

Hubble constant	H_0	=	$70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$
North Ecliptic Pole (J2000.0)	$(\alpha_{\mathrm{E}},\ \delta_{\mathrm{E}})$		$(18^{\mathrm{h}}00^{\mathrm{m}}00^{\mathrm{s}}, +66°33'39'')$
North Galactic Pole (J2000.0)	$(\alpha_{\mathrm{G}},\ \delta_{\mathrm{G}})$		$(12^{\mathrm{h}}51^{\mathrm{m}}26^{\mathrm{s}}, +27°07'42'')$
1 jansky	1 Jy	=	$10^{-26}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$
1 parsec	1 pc	=	3.086×10^{16} m 206 265 au 3.262 ly
1 astronomical unit (au)	1 au	=	1.496×10^{11} m
1 sidereal day	T_{SD}	=	23.93444 h $23^{\mathrm{h}}56^{\mathrm{m}}04^{\mathrm{s}}$
1 tropical year		=	365.2422 solar days
1 sidereal year		=	365.2564 solar days

Gauss's formulae

Spherical law of cosines: $\cos a = \cos b \cos c + \sin b \sin c \cos A$
Spherical law of sines: $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$

Approximations

$(1 + x)^n \approx 1 + nx$
$(1 + x)(1 + y) \approx 1 + x + y$ if $x \ll 1$ and $y \ll 1$

The Sun

Solar luminosity	$L_\odot$	$=$	3.826×10^{26} W
Apparent angular diameter of Sun	$\theta_\odot$	$=$	$32'$
Effective temperature of Sun	$T_{\mathrm{eff},\odot}$	$=$	5778 K
Apparent visual magnitude		$=$	-26.75
Absolute visual magnitude		$=$	+4.82
Apparent bolometric magnitude		$=$	-26.83
Absolute bolometric magnitude		$=$	+4.74
Distance of the Sun from the Galactic centre		$\approx$	8 kpc

The Earth and Moon

Obliquity of the ecliptic (Earth)	ϵ	$=$	$23.5'$
Platonic year (period of precession of Earth's axis)		$=$	25 765 years
Apparent visual magnitude of full Moon		$=$	-12.74
Apparent angular diameter of Moon	θ_{L}	$=$	$31'$
Inclination of the lunar orbit to the ecliptic		$=$	$05°08'43''$
Inclination of the lunar equator to its orbital plane		$=$	6.687°
Lunar sidereal month	T_{SL}	$=$	27.321661 d 655.71986 h
Synodic month		$=$	29.530589 d
Tropical month		$=$	27.321582 d
Anomalistic month		$=$	27.554550 d
Draconic month		$=$	27.212221 d

The Solar System

Object	**Mean radius** [km]	**Mass** [kg]	**Semimajor axis** [au]	**Eccentricity**
Sun	695 700	1.988×10^{30}	—	—
Mercury	2 440	3.301×10^{23}	0.387	0.206
Venus	6 052	4.867×10^{24}	0.723	0.007
Earth	6 378	5.972×10^{24}	1.000	0.016 710
Moon	1 737	7.346×10^{22}	3.844×10^5 km	0.054 900 (range $0.026 - 0.077$)
Mars	3 390	6.417×10^{23}	1.524	0.093
Jupiter	69 911	1.898×10^{27}	5.203	0.048
Saturn	58 232	5.683×10^{26}	9.537	0.054
Uranus	25 362	8.681×10^{25}	19.189	0.047
Neptune	24 622	1.024×10^{26}	30.070	0.009

Theory: instructions

- **Do not touch envelopes until the start of the examination.**
- The theoretical examination lasts for 5 hours and is worth a total of 250 marks.
- There are **Answer Sheets** for carrying out detailed work and **Working Sheets** for rough work, which are already marked with your student code and question number.
- ***Use only the answer sheets for a particular question for your answer. Please write only on the printed side of the sheet. Do not use the reverse side.*** If you have written something on any sheet which you do not want to be evaluated, cross it out.
- Use as many mathematical expressions as you think may help the evaluator to better understand your solutions. The evaluator may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
- You are not allowed to leave your work desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, etc.), please draw the attention of the supervisor.
- The beginning and end of the examination will be indicated by the supervisor. The remaining time will be displayed on a clock.
- At the end of the examination you must stop writing immediately. Put everything back in the envelope and leave it on the table.
- Once all envelopes are collected, your student guide will escort you out of the examination room.
- A list of constants and useful relations are included in the envelope.

Theory 1: 'Neptune'

Given that Neptune will be at opposition on 21 September 2024, calculate in which year Neptune was last at opposition near the time of the northern-hemisphere spring equinox. Assume that the orbits of Earth and Neptune are circular.

(5 points)

Solution

Using Kepler's Third law and the semi-major axis ($a = 30.070$ au) of the orbit from the table of constants, the sidereal period of Neptune's orbit is:

$$P_{\mathrm{N}} = (a^3)^{\frac{1}{2}} = \sqrt{27189.4} = 164.89\,\text{years} \qquad \text{(1 point)}$$

The rest of the calculation can be done in (at least) two ways:

(1) 'day counting'/'date drift' solution:

Since Neptune is an outer planet (relative to Earth), the synodic period S in years is given by:

$$\frac{1}{S} = \frac{1}{1\,\text{year}} - \frac{1}{P_{\mathrm{N}}} \qquad \text{(1 point)}$$

$$= 1 - \frac{1}{164.89} = 1 - 0.006065 = 0.99394$$

$$\implies S = 1/0.99394 = 1.006102\,\text{years} \qquad \text{(1 point)}$$

$$\implies 1.006102 \times 365.2422 = 367.4708\,\text{solar days}$$

Therefore the date of opposition drifts by:

$$D = 367.4708 - 365.2422 = 2.2286\,\text{days/year} \qquad \text{(1 point)}$$

The approximate date of the northern spring equinox is 20 March. The number of days between 21 September and 20 March = 185 days, therefore year of desired opposition is:

$$2024 - (185/D) = 2024 - 83 = 1941 \qquad \text{(1 point)}$$

If the student takes 21 March as the spring equinox, they get $184/D = 82.5\,\text{yr} \implies 1942$.

(2) 'angular drift' solution (more accurate)

As before, the synodic period of Neptune can be derived as:

$$\frac{1}{S} = \frac{1}{1\,\text{year}} - \frac{1}{P_{\mathrm{N}}} \qquad \text{(1 point)}$$

$$= 1 - \frac{1}{164.89} = \frac{163.89}{164.89} \implies S = \frac{164.89}{163.89}\,\text{years}$$

Therefore the ecliptic longitude of Neptune at opposition drifts by $(360°/163.89)$/year. We want to find how many years it takes for the longitude of opposition to move by 180°, i.e. for what t:

$$t \times \frac{360°}{163.89} = 180°. \qquad \text{(2 points)}$$

$$\implies t = 163.89/2 = 81.95\,\text{years} \implies 2024 - 82 = 1942 \qquad \text{(1 point)}$$

We accept 1941 or 1942 for full points for calculations using the assumptions in the question. If the student uses some other method which is conceptually correct and results in 1943 (the true answer) they should also get full points.

Table of spring equinoxes and oppositions of Neptune:

Year	Equinox (UT)	Opposition (UT)	Coordinates of Neptune	ΔT [days]
1940	Mar 20 18:42	Mar 14 21:08	11h40m $+3°28'$	5.9
1941	Mar 21 00:20	Mar 17 07:40	11h49m $+2°39'$	3.7
1942	Mar 21 06:11	Mar 19 18:12	11h57m $+1°50'$	1.5
1943	Mar 21 12:03	Mar 22 04:51	12h04m $+1°00'$	-0.7
1944	Mar 20 17:49	Mar 23 15:29	12h13m $+0°11'$	-2.9
2024	Mar 20 03:06	Sep 21 00:16	23h55m $+1°56'$	-184.9

Taking into account all effects, the last opposition closest to the spring equinox was actually in 1943. 1942 results from the assumptions made in the question.

Theory 2: ‘Magnetic field’

An emission line of wavelength $\lambda = 600$ nm was observed in the spectrum of a white dwarf. Assuming that it originates from the interaction of an electron with a magnetic field,

(a) calculate the magnetic flux density of the field;

(b) estimate the wavelength of another spectral line, the discovery of which could confirm that the lines originate from particles of a plasma interacting with the magnetic field.

(5 points)

Solution

(a) In a magnetic field, a charged particle moves along a circular path defined by the equality of centrifugal and magnetic forces:

$$mv^2/r = evB, \qquad \text{(1 point)}$$

where m is the mass, v velocity, r radius of the circle, e charge of the particle, and B magnetic flux density.

For circular motion, $v = 2\pi r/T$, therefore $T = 2\pi m/eB$. The charged particle moving in harmonic motion (i.e. along the circular path with constant velocity) emits a wave of wavelength $\lambda = cT = 2\pi mc/eB$ and thus $B = 2\pi mc/e\lambda$. (1 point)

Substituting the numerical values, including the mass and charge of the electron:

$$B = (2\pi \times 9.109 \times 10^{-31} \times 2.998 \times 10^{8})/(1.602 \times 10^{-19} \times 6 \times 10^{-7}) \approx 2 \times 10^{4}\ \text{T}. \quad \text{(1 point)}$$

Since λ is given to 1 s.f., the correct answer is 20 kT, however accept 17.9 kT or 18 kT. More than 3 s.f. in the final answer loses points.

(b) In the plasma, besides electrons, only protons will be present in large quantities; protons will emit energy at a wavelength larger in proportion to the mass ratio, i.e. 1836× larger (anything within 1800–2000× $\implies$ λ =1.08–1.20 mm is acceptable).

(2 points)

Theory 3: 'Microlensing'

A faint subdwarf star ($I = 20.4\,\mathrm{mag}$) in the Galactic bulge was observed to brighten to $I' = 15.2\,\mathrm{mag}$ as a result of gravitational microlensing, allowing a high-resolution spectrum to be obtained with the UVES spectrograph on the Very Large Telescope (mirror diameter 8.2 m).

Estimate the diameter of the telescope needed to obtain a spectrum of the same quality with the same instrument and exposure time for this star at its normal apparent brightness.

(5 points)

Solution

Let F be the unmagnified flux of the star. During gravitational microlensing, the apparent flux is magnified by a factor of A, thus using the formula relating magnitude to flux:

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right), \qquad \text{(1 point)}$$

$$I - I' = -2.5 \log_{10}\left(\frac{F}{AF}\right) = 2.5 \log A \qquad \text{(1 point)}$$

and so

$$A = 10^{2.08} \approx 120. \qquad \text{(1 point)}$$

Let D be the effective mirror diameter of a telescope which would collect the same number of photons in unit time from the unbrightened star as VLT from the brightened star. We have:

$$FD^2 = AFD^2_{\mathrm{VLT}}, \qquad \text{(1 point)}$$

therefore:

$$D = \sqrt{A}\, D_{\mathrm{VLT}} \approx 90\,\mathrm{m}. \qquad \text{(1 point)}$$

Theory 4: 'Europa'

(a) Assuming that the ice covering the ocean on Jupiter's moon Europa is 6 km thick, that the surface temperature on the night side of Europa is 100 K and that the temperature at the ice-water boundary is 273 K, calculate the total power corresponding to the heat emitted from the interior of this moon.

(b) On Earth, the geothermal heat flux measured at the surface is $70 \times 10^{-3}\,\mathrm{Wm^{-2}}$ and originates mainly from radioactive decay. Is the heat emanating from the interior of Europa more likely to come from radioactive decay or tidal forces? (Select the correct answer on the answer sheet and show your working.)

(10 points)

Notes: the heat passing through a wall with a surface S and thickness d in time t is described by the formula:

$$Q = \lambda S \Delta T t/d,$$

where λ stands for thermal conductivity and ΔT for the temperature difference.

The thermal conductivity of ice $\lambda = 3\,\mathrm{Wm^{-1}K^{-1}}$. The mass and radius of Europa are 4.8×10^{22} kg and 1561 km.

Solution

(a) From the formula $Q = \lambda S \Delta T t/d$, calculate the power P of the heat flowing through a unit of the surface of ice:

$$P = Q/t = \lambda S \Delta T/d. \qquad \text{(1 point)}$$

Approximating the ice crust of Europa as a 'wall', i.e. that the upper and lower surfaces are of equal area S, we obtain the power per unit surface area: (1 point)

$$P/S = \lambda \Delta T/d.$$

Substituting the data we obtain

$$P/S = 86.5 \times 10^{-3}\,\mathrm{Wm^{-2}} \approx 87\,\mathrm{mWm^{-2}}, \qquad \text{(1 point)}$$

similar to the value given for the Earth.
The total power emitted inside Europa is therefore equal to

$$4\pi R_{Eu}^2\,(P/S) = 2.65 \times 10^{12}\,\mathrm{W}. \qquad \text{(2 points)}$$

(b) The total power emitted inside the Earth is equal to

$$4\pi R_{\oplus}^2 \times 70 \times 10^{-3}\,\mathrm{Wm^{-2}} = 36 \times 10^{12}\,\mathrm{W}, \qquad \text{(1 point)}$$

which is about 13.5× larger than the value for Europa. However, the mass ratio is 124. For the heat in Europa's interior to have a purely radioactive origin, the matter on Europa would have to contain an order of magnitude more radioactive elements per unit mass, which excludes this explanation. (3 points)

Thus the answer must be tidal forces. (1 point)

Theory 5: 'Dark Energy'

Observations indicate that the expansion of the Universe is accelerating. Fluctuations of the cosmic microwave background favour a flat (Euclidean) geometry, in which the total mass density (i.e. density of matter and equivalent mass density of all forms of energy) should be equal to the so-called critical density:

$$\rho_{\rm cr} = \frac{3\,H_0^2}{8\,\pi\,G},$$

where H_0 is the present value of the Hubble constant. However, the total density of matter (luminous and dark) is estimated at

$$\rho_{\rm m,0} \approx 2.8 \cdot 10^{-27}\,{\rm kg\,m^{-3}}\,.$$

To resolve this discrepancy, the standard cosmological model assumes that the Universe is filled with a mysterious 'dark energy' of constant energy density E_Λ.

Determine the value of E_Λ and calculate for which redshift in the past the energy density equivalent to matter was equal to the density of dark energy. Neglect the contribution of electromagnetic radiation.

(12 points)

Solution

Substituting the values of H_0 and G from the table of constants,

$$\rho_{\rm cr} = 9.202 \times 10^{-27}\,{\rm kg\,m^{-3}}. \qquad \text{(1 point)}$$

In flat geometry we have:

$$\rho_{m,0} + \frac{E_\Lambda}{c^2} = \rho_{\rm cr}\,. \qquad \text{(2 points)}$$

Hence E_Λ is given by:

$$E_\Lambda = [\rho_{\rm cr} - \rho_{m,0}]\,c^2 \approx 5.756 \times 10^{-10}\,{\rm J\,m^{-3}}. \qquad \text{(1 points)}$$

The linear scale of the Universe, a, is related to the cosmological redshift:

$$a(z) = a_0/(1+z)\,. \qquad \text{(2 points)}$$

Thus, the matter density, ρ_m, increases with redshift:

$$\rho_m(z) = \rho_{m,0}\,(1+z)^3. \qquad \text{(2 points)}$$

We substitute the matter density, ρ_m by the energy density, E_m, using the relationship $E = mc^2$

$$E_m(z) = \rho_m(z)\,c^2 = \rho_{m,0}\,(1+z)^3\,c^2. \qquad \text{(2 points)}$$

We finally get

$$z_{eq} = \left[\frac{E_\Lambda}{\rho_{m,0}\,c^2}\right]^{1/3} - 1 \approx 0.32\,. \qquad \text{(2 points)}$$