Theory 11: 'X-ray emission from galaxy clusters'

Clusters of galaxies are strong X-ray sources. It has been established that the emission mechanism is thermal bremsstrahlung (free-free radiation) from a hot hydrogen and helium plasma inside the cluster. The luminosity L_X (in Watts) of each component of the plasma is described by the formula:

$$L_X = 6 \times 10^{-41} N_e N_X T^{\frac{1}{2}} V Z_X^2,$$

where the symbols represent:

$X-$ Hydrogen (H) or Helium (He),
N_e- number density of electrons [m^{-3}],
N_X- number density of ions X [m^{-3}],
Z_X- atomic number of ion X,
$T-$ temperature of the plasma [K],
$V-$ volume occupied by the plasma [m^3].

(a) Determine the total mass (in solar masses) of the plasma which emits the X-rays, assuming that:

- the plasma is fully ionized with 1 helium ion for every 10 hydrogen ions;
- $L_{\text{total}} = 1.0 \times 10^{37}$ W,
- $T = 80 \times 10^6$ K,
- the plasma is uniformly distributed in a sphere of radius $R = 500$ kpc,
- self-absorption is negligible.

(16 points)

The photons of the cosmic microwave background (CMB) interact with plasma in a process known as inverse Compton scattering. The CMB normally has a thermal blackbody spectrum at a temperature of 2.73 K. However, interaction with the plasma leads to distortion of the CMB spectrum (known as the Sunyaev–Zeldovich effect).

(b) Estimate the mean free path of CMB photons in the plasma, i.e. the average distance travelled by a photon before interacting with an electron. Express it in Mpc. The effective cross section for photon–electron interactions is $\sigma = 6.65 \times 10^{-29}$ m^2. (5 points)

(c) Estimate the typical energy of CMB photons. (3 points)

(d) The energy of CMB photons can be increased by a factor of up to $(1 + \beta)/(1 - \beta)$ due to the inverse Compton scattering, where $v = \beta c$ is the velocity of electrons. Estimate the energy of scattered CMB photons. (6 points)

(Total: 30 points)

Solution

Part (a)

Concentrations of electrons, N_e, and He nuclei, N_{He}, is related to the concentration of H nuclei, N_{H}:

$$N_{\mathrm{He}} = 0.1\, N_{\mathrm{H}} \qquad \text{(1 point)}$$
$$N_e = N_{\mathrm{H}} + 2\, N_{\mathrm{He}} = 1.2\, N_{\mathrm{H}} \qquad \text{(2 points)}$$

The total X-ray emission is a sum of the bremmstrahlung generated by interaction of electrons with H and He nuclei:

$$L = 6 \cdot 10^{-41}\, N_e\, T^{\frac{1}{2}}\, V\, (N_{\mathrm{H}} + Z_{\mathrm{He}}^2\, N_{\mathrm{He}}),$$

where $Z_{\mathrm{He}} = 2$. The luminosity L is expressed by the concentration of H nuclei:

$$L = 6 \cdot 10^{-41}\, 1.2\, N_{\mathrm{H}}\, T^{\frac{1}{2}}\, V\, 1.4\, N_{\mathrm{H}},$$
$$\Longrightarrow L = (10.08 \times 10^{-41})\, T^{\frac{1}{2}}\, V\, N_{\mathrm{H}}^2 \approx 10^{-40}\, T^{\frac{1}{2}}\, V\, N_{\mathrm{H}}^2. \qquad \text{(3 points)}$$

The volume V is given by:

$$V = \frac{4}{3}\pi R^3 = 1.54 \cdot 10^{67}\ \mathrm{m}^3. \qquad \text{(3 points)}$$

Thus, the concentration of N_{H}:

$$N_{\mathrm{H}} = \left(\frac{L}{10^{-40}\, T^{\frac{1}{2}}\, V}\right)^{1/2} \approx 8.48 \times 10^2\ \mathrm{m}^{-3}. \qquad \text{(3 points)}$$

To obtain the total mass of the plasma, M, one should multiply the volume V by the the sum of H and He mass densities:

$$M = V\, (N_{\mathrm{H}}\, m_{\mathrm{H}} + N_{\mathrm{He}}\, m_{\mathrm{He}}) = 3.03 \times 10^{43}\,\mathrm{kg} \approx 1.52 \times 10^{13}\,\mathrm{M}_{\odot}. \qquad \text{(4 points)}$$

where m_{H} and m_{He} are the masses of hydrogen and helium atoms.

Part (b)

Let L be the mean free path of a photon. The number of electrons in a cylinder with a cross section area of σ and a length of L equals

$$N = nL\sigma = 1 \implies L = 1/(n\sigma).$$

Assuming $n = N_e = 1.2N_{\mathrm{H}} = 1.018 \times 10^3\,\mathrm{m}^{-3}$, we get

$$L = \frac{1}{1.018 \times 10^3 \cdot 6.65 \times 10^{-29}} = 1.48 \times 10^{25}\,\mathrm{m} = 4.79 \times 10^8\,\mathrm{pc} \approx 500\,\mathrm{Mpc}. \qquad \text{(5 points)}$$

(The radius of the cluster is ~ 1000 times smaller than L, so the interactions between the CMB photons and hot electrons are rare.)

Part (c)

Using Wien's law,

$$\lambda = \frac{b}{T} = \frac{2.898 \times 10^{-3}}{2.73} = 1.07 \times 10^{-3}\ \mathrm{m}$$
$$\Longrightarrow E_0 = \frac{hc}{\lambda} = 1.85 \times 10^{-22}\ \mathrm{J} \qquad \text{(4 points)}$$

Part (d)

We first have to estimate the typical velocity v_e of electrons using the formula for the kinetic energy of the particles in a gas.

$$\frac{1}{2}mv^2 = \frac{3}{2}kT \implies v_e = \sqrt{\frac{3kT}{m_e}}$$

$$\therefore v_e = \sqrt{\frac{3 \cdot 1.381 \times 10^{-23} \cdot 8 \times 10^7}{9.109 \times 10^{-31}}} = 6.032 \times 10^7 \ \mathrm{m\,s^{-1}} = 0.202c$$

(3 points)

The energy of upscattered photons is:

$$E' = \frac{1+\beta}{1-\beta}E_0 \approx 1.5E_0 = 2.78 \times 10^{-22} \text{ J.}$$ (2 points)

Note: The formula for the kinetic energy of particles in an ideal gas remains valid even at such high temperatures. This is because

$$x = \frac{m_e c^2}{kT} = \frac{9.109 \times 10^{-31} \cdot (2.998 \times 10^8)^2}{1.381 \times 10^{-23} \cdot 8 \times 10^7} = 74.1 \gg 1,$$

and the electrons can be treated as non-relativistic. The full relativistic formula for the rms velocity of particles in an ideal gas is

$$< v_e^2 >= \frac{xc^2}{K_2(x)} \int_0^\infty \frac{\sinh^4 \phi}{\cosh \phi} e^{-x \cosh \phi} d\phi,$$

where $K_2(x)$ is a modified Bessel function of the second kind. It can be shown that for $x = 74.1$ this formula yields $\sqrt{< v_e^2 >} = 0.198c$, which is virtually identical to the prediction of the non-relativistic formula.

Theory 12: 'DART'

The Double Asteroid Redirection Test (DART) was a NASA mission to evaluate a method of planetary defense against near-Earth objects. The spacecraft hit Dimorphos, a moon of the asteroid Didymos, to study how the impact affected its orbit.

(a) Calculate the expected orbital period change (in minutes), assuming that the collision was head-on, central, and perfectly inelastic.

Assume that before the impact Dimorphos orbited Didymos on a circular orbit with a period of $P = 11.92$ h. The masses of Dimorphos and Didymos are $m = 4.3 \times 10^9$ kg and $M = 5.6 \times 10^{11}$ kg, respectively. The mass and speed of the DART spacecraft relative to Dimorphos at a moment of impact were $m_\mathrm{s} = 580\,\mathrm{kg}$ and $v_\mathrm{s} = 6.1\,\mathrm{km\,s^{-1}}$. Neglect the gravitational influence of other bodies.

(20 points)

(b) In reality, the orbital period of Dimorphos was observed to be changed by $\Delta P_0 = -33\,\mathrm{min}$. This is due to the momentum transfer associated with the recoil of the ejected debris: the spacecraft was absorbed by the asteroid, but the impact excavated some material from the asteroid and ejected it into space. Calculate the momentum of the ejected debris and express it as a fraction of the momentum of Dimorphos before the collision. You can assume that the mass of the ejected material is much smaller than the mass of Dimorphos.

(15 points)

(c) Calculate the velocity change (in $\mathrm{mm\,s^{-1}}$) of Dimorphos as a result of the impact, taking into account the effect of the ejected debris.

(5 points)

(Total: 40 points)

Solution

Part (a)

Didymos' mass is much larger than Dimorphos' mass. Therefore, the radius of the orbit of Dimorphos before impact, a, can be calculated from Kepler's 3rd law:

$$\frac{GM}{4\pi^2} = \frac{a^3}{P^2},$$

and therefore:

$$a = \left(\frac{GMP^2}{4\pi^2}\right)^{1/3} = 1.2\,\text{km}.$$

The orbital velocity of Dimorphos before impact was:

$$v_0 = \frac{2\pi a}{P} = 0.176\,\text{m/s}. \qquad \text{(2 points)}$$

Let v' be the Dimorphos velocity right after the collision. Using the law of conservation of momentum, we have:

$$mv_0 - m_s v_s = (m + m_s)v',$$

and:

$$v' = \frac{mv_0 - m_s v_s}{m + m_s} \approx v_0 - \frac{m_s}{m} v_s,$$

where we used the fact that the mass of the spacecraft mass is much smaller than the mass of Dimorphos.

We then use the vis-viva equation to calculate the semi-major axis of the orbit after collision a':

$$(v')^2 = GM\left(\frac{2}{a} - \frac{1}{a'}\right) = \frac{GM}{a}\left(2 - \frac{a}{a'}\right) = v_0^2\left(2 - \frac{a}{a'}\right),$$

so

$$2 - \frac{a}{a'} = \left(\frac{v'}{v_0}\right)^2 = \left(1 - \frac{m_s}{m}\frac{v_s}{v_0}\right)^2 \approx 1 - \frac{2m_s}{m}\frac{v_s}{v_0}.$$

Thus, the semi-major axis changed by:

$$\frac{\Delta a}{a} = \frac{a' - a}{a} = -\frac{2m_s}{m}\frac{v_s}{v_0}. \qquad \text{(8 points)}$$

If the semi-major axis changes from a to $a + \Delta a$, then the orbital period changes from P to $P + \Delta P$, and the mass of the spacecraft can be neglected. Then:

$$\frac{a^3}{P^2} = \frac{(a + \Delta a)^3}{(P + \Delta P)^2} = \frac{a^3(1 + \Delta a/a)^3}{P^2(1 + \Delta P/P)^2} = \frac{a^3}{P^2}\left(1 + \frac{3\Delta a}{a} - \frac{2\Delta P}{P}\right),$$

hence:

$$\frac{\Delta P}{P} = \frac{3}{2}\frac{\Delta a}{a}.$$

Thus, the orbital period changes by:

$$\frac{\Delta P}{P} = \frac{3}{2}\frac{\Delta a}{a} = -\frac{3m_s}{m}\frac{v_s}{v_0} \qquad \text{(8 points)}$$

We therefore expect that the orbital period of Dimorphos should decrease by 1.4%, that is, 10 minutes. (2 points)

Alternative solution (requires better numerical precision)

$$v_0 = 0.17622 \text{ m/s}$$

$$v' = 0.17662 - 0.00082 = 0.17540 \text{ m/s}$$

$$a' = 0.99080a = 1192.5 \text{ m}$$

$$P' = 705.4 \text{ min}$$

Hence $\Delta P = 705.4 - 715.2 = -9.8 \approx -10$ min. I propose to grant full points ONLY if the first four significant figures match the solution. If the first three significant figures match the solution, grant 80% of the points. Otherwise (if the method is correct), grant HALF of the points.

Part (b)

Let Δp be the momentum of the ejected debris. Then, the momentum conservation equation becomes:

$$mv_0 - m_s v_s = (m + m_s)v' + \Delta p, \qquad \text{(4 points)}$$

so:

$$v' = \frac{mv_0 - m_s v_s - \Delta p}{m + m_s} \approx v_0 - \frac{m_s}{m}v_s - \frac{\Delta p}{m}.$$

Using similar calculations as in point a), we get:

$$\frac{\Delta a}{a} = -2\left(\frac{m_s}{m}\frac{v_s}{v_0} + \frac{\Delta p}{mv_0}\right), \qquad \text{(8 points)}$$

hence:

$$\frac{\Delta P_0}{P} = \frac{3}{2}\frac{\Delta a}{a} = -3\left(\frac{m_s}{m}\frac{v_s}{v_0} + \frac{\Delta p}{mv_0}\right).$$

Thus:

$$\frac{\Delta p}{mv_0} = -\frac{\Delta P_0}{3P} - \frac{m_s}{m}\frac{v_s}{v_0} = 0.011. \qquad \text{(3 points)}$$

Alternative solution (requires better numerical precision)

The orbital period after the collision is $P' = 11.92 - 33/60 = 11.37$ h. Thus, the semi-major axis of the orbit (after the collision) is

$$a' = \left(\frac{GMP'^2}{4\pi^2}\right)^{1/3} = 1166\,\text{m}.$$

The velocity of Dimorphos right after the collision is $v' = v_0\sqrt{2 - a/a'} = 0.1734\,\text{m}\,\text{s}^{-1}$, so $\Delta p/mv_0 = 0.011$. I propose to grant full points ONLY if the first four significant figures match the solution. If the first three significant figures match the solution, grant 80% of the points. Otherwise (if the method is correct), grant HALF of the points.

Part (c)

$$\Delta v = v' - v_0 = -\frac{m_s}{m}v_s - \frac{\Delta p}{m} = -\frac{m_s}{m}v_s + \frac{\Delta P_0}{3P}v_0 + \frac{m_s}{m}v_s = \frac{\Delta P_0}{3P}v_0 = -2.7\,\text{mm/s}.$$

(5 points)

Theory 13: 'LISA'

The Laser Interferometer Space Antenna (LISA) is a proposed experiment to detect low-frequency gravitational waves. It consists of three spacecraft arranged in an equilateral triangle. A passing gravitational wave changes the distance between the spacecraft, which can be precisely measured (more details are given in the notes below).

One of the sources of low-frequency gravitational waves are compact binary star systems, for example binary white dwarfs. Such a system was recently discovered at a distance of 2.34 kpc from the Sun. The orbital period of the binary was found to be 414.79 s and is changing at a rate of $-7.49 \times 10^{-4}\,\mathrm{s\,yr^{-1}}$ due to the emission of gravitational waves.

(a) Check if this binary system can be detected by LISA. (25 points)

(b) Calculate the chirp mass. (5 points)

(c) Determine the masses of both components knowing that the ratio between the radius of one of the components to the semi-major axis of the orbit is 0.139, and assuming both components follow the mass–radius relation for white dwarfs given in the table below. (15 points)

(Total: 45 points)

Notes:

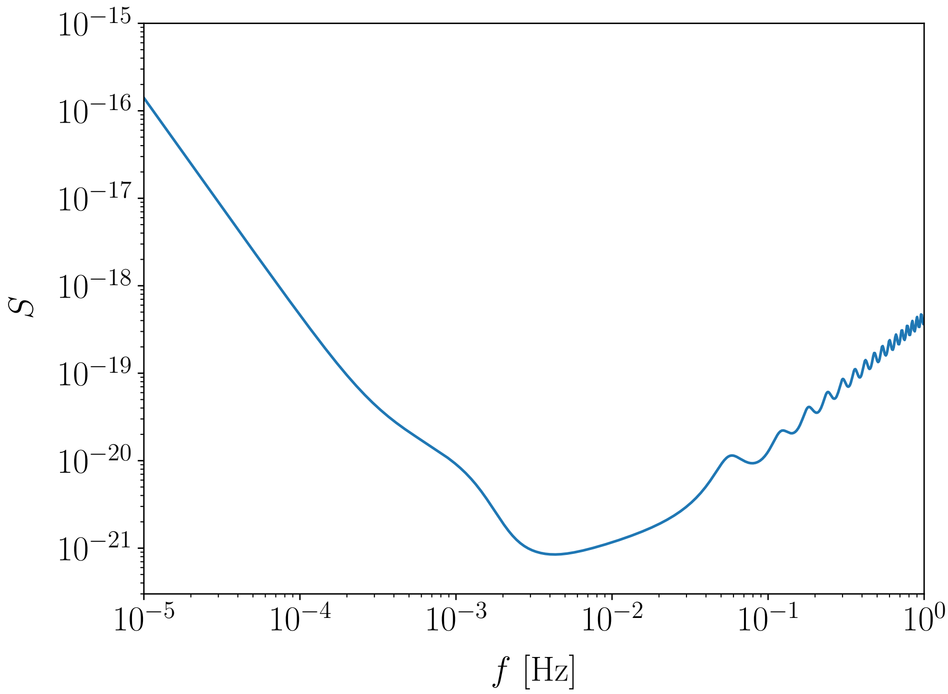
1. A binary star system with an orbital period P emits gravitational waves with a frequency of $f = 2/P$.

2. LISA measures a dimensionless quantity called the characteristic strain amplitude, S, given by
$$S = h\sqrt{f T_{\mathrm{obs}}},$$
where $T_{\mathrm{obs}} = 4\,\mathrm{yr}$ is the expected duration of the mission. h is the gravitational wave strain, given by:
$$h = \frac{2(G\mathcal{M})^{5/3}(\pi f)^{2/3}}{c^4 D},$$
where $\mathcal{M}$ is the so-called chirp mass, f is the frequency of the gravitational wave and D is the distance to the system. If we denote the masses of the components of the binary as M_1 and M_2, then the chirp mass is given by:
$$\mathcal{M} = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}}.$$
The expected sensitivity of LISA as a function of a gravitational wave frequency is presented on the figure below.

3. The semi-major axis a of the binary system changes due to the emission of gravitational waves at a rate:
$$\frac{\Delta a}{\Delta t} = -\frac{64}{5}\frac{G^3}{c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^3}.$$

M ($M_\odot$)	R ($R_\odot$)
0.48	0.0144
0.50	0.0147
0.52	0.0150
0.54	0.0153
0.56	0.0156
0.58	0.0159
0.60	0.0162
0.62	0.0165
0.64	0.0168

Mass–radius relation for white dwarfs based on theoretical models of Althaus et al. (2013) for white dwarfs of $log_g = 7.7$.



The expected sensitivity of LISA as a function of gravitational wave frequency.

Solution

Part (a)

To determine whether the system can be detected by LISA, we need to determine two quantities: the gravitational-wave frequency and the characteristic strain amplitude.

It is straightforward to calculate the gravitational-wave frequency:

$$f = \frac{2}{P} = \frac{2}{414.79} = 4.8 \times 10^{-3}\,\mathrm{Hz}.$$

This frequency is within the LISA band and close to the maximum LISA sensitivity.

Calculating gravitational wave frequency	2 points

To estimate the characteristic strain amplitude, we need to know the chirp mass $\mathcal{M}$. The other required quantities (such as the gravitational wave frequency, distance, and duration of the LISA observations) are already known.

The orbital period change rate $\Delta P/\Delta t$ is given in the problem. We need to link it to $\Delta a/\Delta t$ which we know from Note 3 is linked to the mass function. From Kepler's third law, we know that:

$$\frac{a^3}{P^2} = \frac{G(M_1 + M_2)}{4\pi^2},$$

where M_1 and M_2 are masses of both components of the system. If, due to the emission of gravitational waves, the semi-major axis changes from a to $a + \Delta a$, then the orbital period changes from P to $P + \Delta P$, as the masses of both components are constant. Then:

$$\frac{a^3}{P^2} = \frac{(a+\Delta a)^3}{(P+\Delta P)^2} = \frac{a^3(1+\Delta a/a)^3}{P^2(1+\Delta P/P)^2} = \frac{a^3}{P^2}\left(1 + \frac{3\Delta a}{a} - \frac{2\Delta P}{P}\right),$$

hence:

$$\frac{\Delta P}{P} = \frac{3}{2}\frac{\Delta a}{a}.$$

Here, we used the fact that $(1+x)^n \approx 1 + nx$ for $x \ll 1$.

Therefore:

$$\begin{aligned}
\frac{\Delta P}{\Delta t} &= \frac{3P}{2a}\frac{\Delta a}{\Delta t} = -\frac{3}{2}\cdot\frac{64}{5}\frac{P}{a}\frac{G^3}{c^5}\frac{M_1 M_2(M_1+M_2)}{a^3} = -\frac{96}{5}\frac{G^3}{c^5}\frac{P}{a}\frac{M_1M_2(M_1+M_2)}{G(M_1+M_2)P^2}\cdot 4\pi^2 \\
&= -\frac{96}{5}(2\pi)^2\frac{G^2}{c^5}\frac{M_1M_2}{aP} = -\frac{96}{5}(2\pi)^2\frac{G^2}{c^5}\frac{M_1M_2}{P}\frac{(2\pi)^{2/3}}{G^{1/3}(M_1+M_2)^{1/3}P^{2/3}} \\
&= -\frac{96}{5}(2\pi)^{8/3}\frac{G^{5/3}}{c^5}\frac{M_1M_2}{(M_1+M_2)^{1/3}}\frac{1}{P^{5/3}} = -\frac{96}{5}(2\pi)^{8/3}\frac{G^{5/3}}{c^5}\frac{M_1M_2}{(M_1+M_2)^{1/3}}\frac{1}{P^{5/3}} \\
&= -\frac{96}{5c^5}(2\pi)^{8/3}\left(\frac{G\mathcal{M}}{P}\right)^{5/3} = -\frac{192\pi}{5c^5}(G\mathcal{M})^{5/3}\left(\frac{P}{2\pi}\right)^{-5/3},
\end{aligned}$$

Thus, by knowing the orbital period and its rate of change from observations, we can determine the chirp mass:

$$\mathcal{M} = \left(\frac{5}{192\pi}\right)^{3/5}\frac{c^3}{G}\frac{P}{2\pi}\left(-\frac{\Delta P}{\Delta t}\right)^{3/5} = 0.319\,M_\odot,$$

and find the characteristic strain amplitude h.

Alternatively, if we notice that the characteristic strain amplitude h depends on $(G\mathcal{M})^{5/3}$ which we can get directly from the rate of change of the period:

$$(G\mathcal{M})^{5/3} = -\frac{\Delta P}{\Delta t}\left(\frac{P}{2\pi}\right)^{5/3}\frac{5c^5}{192\pi},$$

we can skip calculating the chirp mass and get the characteristic strain amplitude directly, which is all we need to check if the binary can be detected.

Either way, the gravitational wave strain is:

$$h = -\frac{5c}{192\pi^2}P\frac{\Delta P}{\Delta t}\frac{1}{D}.$$

If we plug in the numerical values, we get $h = 1.1 \times 10^{-22}$ and $S = 8.4 \times 10^{-20}$. Checking the plot, this is above the expected sensitivity of LISA at 5 mHz. Thus, this object should be detected by LISA.

Calculating the chirp mass or $(G\mathcal{M})^{5/3}$ as a function of P and $\Delta P/\Delta t$	15 points
Calculating h and S	5 points
Correct conclusion – the system may be detected with LISA	3 points

Part (b)

To determine the masses of both components M_1 and M_2 we need two simultaneous $M_1 - M_2$ relations. The expression for the chirp mass:

$$\mathcal{M} = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}},$$

gives us one relation, and we can calculate the chirp mass of the observed system from the rate of change of the period, if that was not already done in part (a):

$$\mathcal{M} = \left(\frac{5}{192\pi}\right)^{3/5}\frac{c^3}{G}\frac{P}{2\pi}\left(-\frac{\Delta P}{\Delta t}\right)^{3/5} = 0.319\,M_\odot.$$

The second relation can be obtained from the mass–radius relation for white dwarfs given in the table and Kepler's third law.

We are told that for one of the components (call it '1'),

$$a = \frac{R_1}{0.139}.$$

From Kepler's third law we know that:

$$M_1 + M_2 = \left(\frac{a}{1\,\mathrm{au}}\right)^3\left(\frac{P}{1\,\mathrm{yr}}\right)^2,$$

which will give us the mass of the second component M_2 from the mass of the first, M_1.

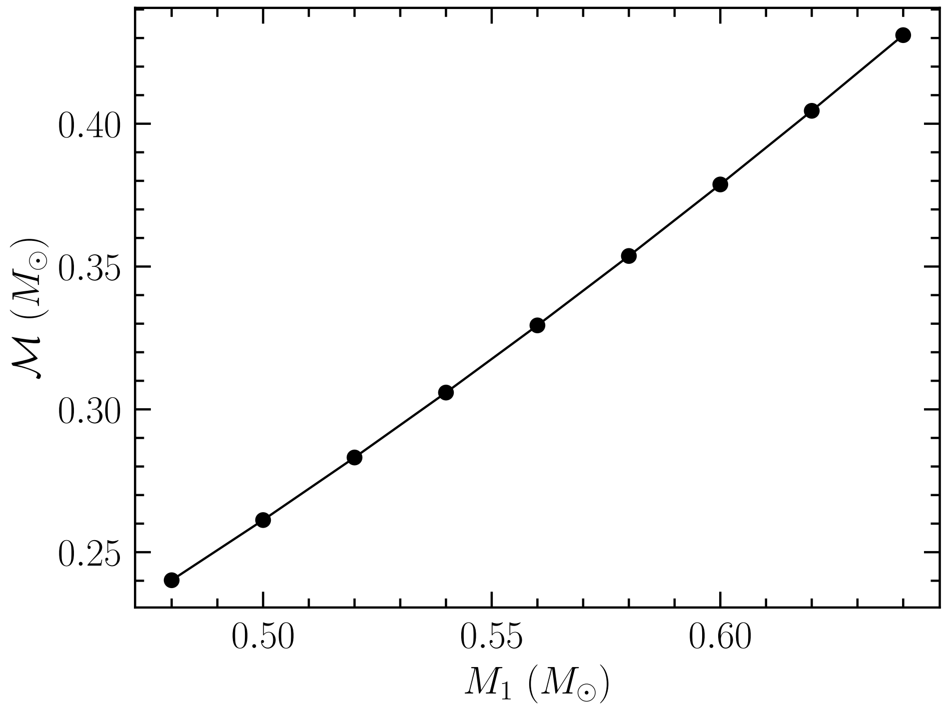
From here, it is not possible to derive an analytical formula for the masses of the components. Instead, we need to use numerical methods to estimate the result.

Taking the masses and radii listed in the given table as M_1 and R_1, we can obtain a and thus $M_1 + M_2$, M_2 and finally $\mathcal{M}$ for each mass:

M_1 ($M_\odot$)	R ($R_\odot$)	a ($R_\odot$)	$M_1 + M_2$	M_2	$\mathcal{M}$
0.48	0.0144	0.104	0.647	0.167	0.240
0.50	0.0147	0.106	0.689	0.189	0.261
0.52	0.0150	0.108	0.732	0.212	0.283
0.54	0.0153	0.110	0.776	0.236	0.306
0.56	0.0156	0.112	0.823	0.263	0.329
0.58	0.0159	0.114	0.871	0.291	0.354
0.60	0.0162	0.117	0.922	0.322	0.379
0.62	0.0165	0.119	0.974	0.354	0.405
0.64	0.0168	0.121	1.028	0.388	0.431

The actual chirp mass is $\mathcal{M} = 0.319\,M_\odot$. Therefore, by linear interpolation or graphically, we estimate $M_1 = 0.55\,M_\odot$ and $M_2 = 0.25\,M_\odot$.
(The student does not need to calculate $\mathcal{M}$ for all values of M_1.)

Graphical solution:



Calculating the chirp mass	5 points
Deriving two $M_1 - M_2$ relations	5 points
Determining the masses of both components	10 points