

SHORT QUESTIONS: MARKING SCHEME

1. Several exoplanets have been observed in the Gliese 876 system ($M_G = 0.33 \pm 0.03M_\odot$) as given in the following table,

Gliese System	Mass	Semi Major Axis (AU)
Gliese 876 b	$2.276 M_J$	0.2083
Gliese 876 c	$0.714 M_J$	0.1296
Gliese 876 d	$6.8 M_\oplus$	0.0208
Gliese 876 e	$15 M_\oplus$	0.334

Where M_\odot is mass of Sun, M_J is mass of Jupiter ($M_J = 1.89813 \times 10^{27} \text{ kg}$), and M_\oplus is mass of Earth. Assume that all these planets revolve around Gliese 876 in the same direction. Two planets are said to be in resonant orbits if the synodic period of one planet with respect to the other planet is an integer multiple of the orbital period of the second planet.

Find if any of the exoplanets of Gliese 876 system may have resonant orbits.

Answer and Marking Scheme:

1	<p>Determine planet periods</p> <p>Synodic period of planet 1 and 2 (assume $P_1 < P_2$) is $\frac{1}{P_s} = \frac{1}{P_1} - \frac{1}{P_2} \rightarrow P_s = \frac{P_1 P_2}{P_2 - P_1}$</p> <p>Resonant orbit happens if</p> $P_s = mP_1 \rightarrow \frac{P_1 P_2}{P_2 - P_1} = mP_1 \rightarrow m = \frac{P_2}{P_2 - P_1} > 1, \text{ integer}$	50								
<table border="1" style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="width: 25%;">Gliese System</th> <th style="width: 25%;">Mass (kg)</th> <th style="width: 25%;">Semi Major Axis (m)</th> <th style="width: 25%;">Period (Earth, days)</th> </tr> </thead> <tbody> <tr> <td></td> <td></td> <td></td> <td style="text-align: center;">$P = 2\pi \sqrt{\frac{a^3}{GM_G}}$</td> </tr> </tbody> </table>			Gliese System	Mass (kg)	Semi Major Axis (m)	Period (Earth, days)				$P = 2\pi \sqrt{\frac{a^3}{GM_G}}$
Gliese System	Mass (kg)	Semi Major Axis (m)	Period (Earth, days)							
			$P = 2\pi \sqrt{\frac{a^3}{GM_G}}$							
Using the data of the Gliese system we have										

	Gliese 876 star	6.64359×10^{29}			
	Gliese 876 b	4.31938×10^{27}	3.1161×10^{10}		60.07568
	Gliese 876 c	1.35564×10^{27}	1.9388×10^{10}		29.48304
	Gliese 876 d	4.079×10^{25}	3.1116×10^9		1.8956
	Gliese 876 e	8.7194×10^{25}	5.0011×10^{10}		122.1432
2	<p>Calculate resonant orbit for each planet From the results above we see that</p> <ul style="list-style-type: none"> Gliese 876 c and Gliese 876 b have resonant orbit $m = \frac{P_b}{P_b - P_c} = 1.96 \approx 2$ <ul style="list-style-type: none"> Gliese 876 b and Gliese 876 e have resonant orbit $m = \frac{P_e}{P_e - P_b} = 1.9679 \approx 2$				<p>25</p> <p>25</p>

2. Volcanic activity on Io, which has a synchronous orbital period, was proposed to be the result of tidal heating mainly from Jupiter. The resultant tidal force on a body is the difference in gravitational force experienced by the near and far sides of that body due to another body. Measurements of the surface distortion of Io via satellite radar altimeter mapping indicate that the surface rises and falls by up to 100m during one-half orbit. Only the surface layers will move by this amount. Interior layers within Io will move by a smaller amount, and thus we assume that on average the entire mass of Io is moved through 50m. Calculate the *average power* of the tidal heating on Io.

Hint: you can use the following approximation $(1+x)^n \approx 1+nx$ for small x .

The mass of Io is $m_{Io} = 8.931938 \times 10^{22}$ kg.

The perijove distance is $r_{peri} = 420000$ km

The apojove distance is $r_{apo} = 423400$ km.

The orbital period of Io is 152853 s.

The radius of Io is $R_{Io} = 1821.6$ km.

Answer and Marking Scheme:

1	<p>Calculate the tidal force</p> <p>The tidal force,</p> $F_{tidal} = F(r - R_{Io}) - F(r + R_{Io})$ $F_{tidal} = -\frac{GM_J m_{Io}}{(r - R_{Io})^2} + \frac{GM_J m_{Io}}{(r + R_{Io})^2}$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2 \left(1 - \frac{R_{Io}}{r}\right)^2} + \frac{GM_J m_{Io}}{r^2 \left(1 + \frac{R_{Io}}{r}\right)^2}$	<p>Correction: The solution should use the elemental mass of Io. It will be corrected accordingly.</p>
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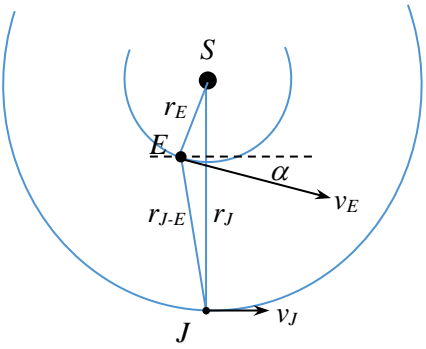
	$F_{tidal} = -\frac{GM_J m_{Io}}{r^2} \frac{1}{\left(1 - \frac{R_{Io}}{r}\right)^2} + \frac{GM_J m_{Io}}{r^2} \frac{1}{\left(1 + \frac{R_{Io}}{r}\right)^2}$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2} \left[\frac{1}{\left(1 - \frac{R_{Io}}{r}\right)^2} - \frac{1}{\left(1 + \frac{R_{Io}}{r}\right)^2} \right]$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2} \left[\frac{1}{\left(1 - 2\frac{R_{Io}}{r}\right)} - \frac{1}{\left(1 + 2\frac{R_{Io}}{r}\right)} \right]$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2} \frac{\left(1 + 2\frac{R_{Io}}{r}\right) - \left(1 - 2\frac{R_{Io}}{r}\right)}{\left(1 - 2\frac{R_{Io}}{r}\right)\left(1 + 2\frac{R_{Io}}{r}\right)}$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2} \frac{\left(1 + 2\frac{R_{Io}}{r} - 1 + 2\frac{R_{Io}}{r}\right)}{\left(1 - 4\left(\frac{R_{Io}}{r}\right)^2\right)}$ $F_{tidal} = -\frac{GM_J m_{Io}}{r^2} \frac{\left(4\frac{R_{Io}}{r}\right)}{\left(1 - 4\left(\frac{R_{Io}}{r}\right)^2\right)}$ <p>The term $\left(\frac{R_{Io}}{r}\right)^2$ is too small and it can be ignored, so that</p> $F_{tidal} = -4 \frac{GM_J m_{Io} R_{Io}}{r^3}$	40
2	<p>Calculate the difference of the tidal force The difference in the tidal force experienced at perijove (closest to Jupiter) and apojove</p>	10

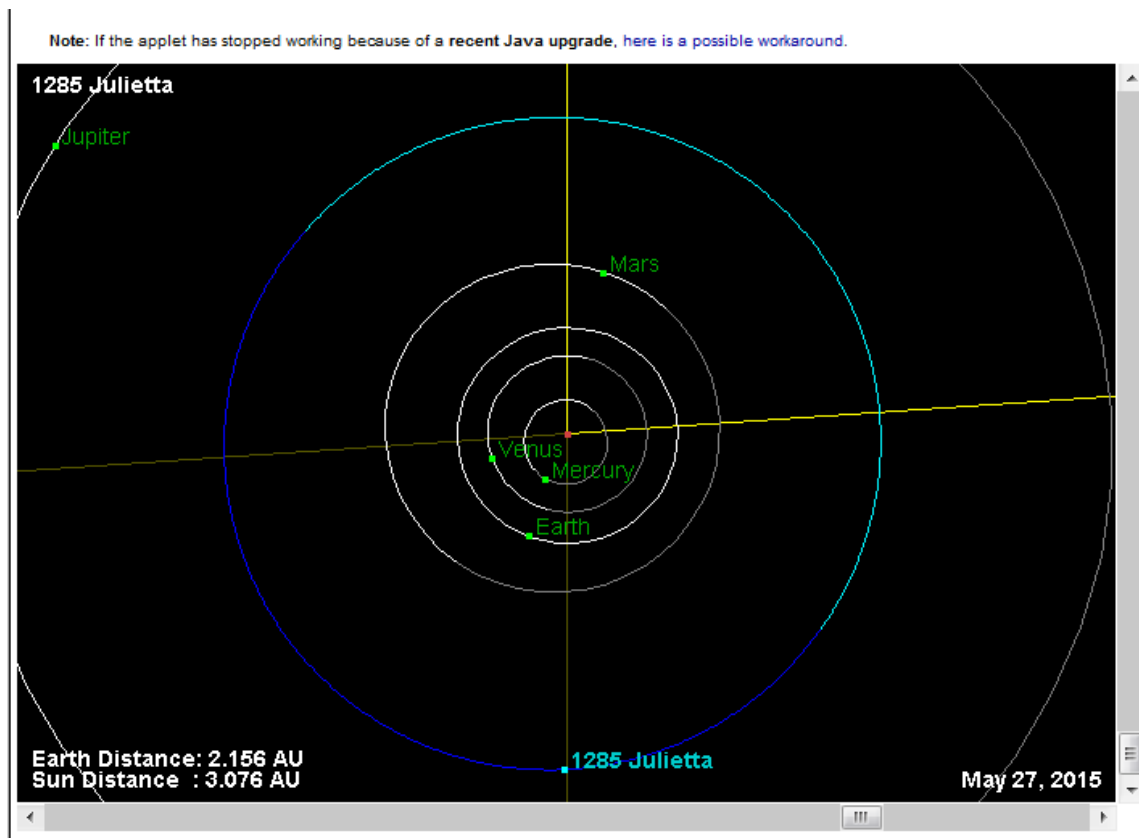
	(furthest from Jupiter) $\Delta F = F_{tidal}(r_{peri}) - F_{tidal}(r_{apo}) $ $= 4GM_J m_{Io} R_{Io} \left[\frac{1}{r_{peri}^3} - \frac{1}{r_{apo}^3} \right] = 2.66 \times 10^{19} \text{ N}$	15 15
3	Calculate the average power of work done Hence, the average power of the work done on the rock during one-half orbit is $\bar{P} = \frac{\Delta F \cdot d}{t_{Io}/2}$ $= 1.74 \times 10^{16} \text{ W}$	10 10

3. On 27 May 2015 at 02:18:49, the occultation of the star HIP 89931 (δ –Sgr) by the asteroid 1285 Julietta was observed from Borobudur temple, which was located at the center of the asteroid shadow path. It lasted for only 6.201 s. Assume that Earth's orbit is circular and the orbit of Julietta is on the ecliptic plane and revolves in the same direction as Earth. At the occultation, Julietta is near its aphelion. At the time of the occultation the distances of Julietta from the Sun and the Earth are 3.076 AU and 2.156 AU respectively. Find the approximate diameter of asteroid Julietta, if the semi major axis of Julietta is $a = 2.9914$ AU.

Answer and Marking Scheme:

1	Calculate the Earth revolution speed Earth revolution speed is $v_E = \frac{2\pi\bar{r}_E}{T_E} = \frac{2\pi a_E}{T_E} = 29805 \text{ m/s}$	20
2	Calculate Julietta revolution speed Julietta revolution is (since the given data is semimajor axis, the orbit of Julietta is ellipse) $v_J = \sqrt{GM \left(\frac{2}{r_J} - \frac{1}{a} \right)} = 16742,9 \text{ m/s}$	25

3	<p>Calculate the approximate dimension of Julietta</p> 		<p>From the geometry</p> $\cos \alpha = \frac{r_E^2 + r_J^2 - r_{J-E}^2}{2r_E r_J} = 0.995$ <p>Thus, Julietta's relative projected speed to Earth is</p> $v_{J,rel} = v_E - v_J \cos \alpha = 13984.16 \text{ m/s}$ <p>Hence, the approximate dimension of Julietta is</p> $l_J = v_{J,rel} t_{occil} = 86701.8 \text{ m}$	<p>40</p> <p>15</p>
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<http://ssd.jpl.nasa.gov/sbdb.cgi?sstr=1285%20julietta;old=0;orb=1;cov=0;log=0;cad=0#orb>

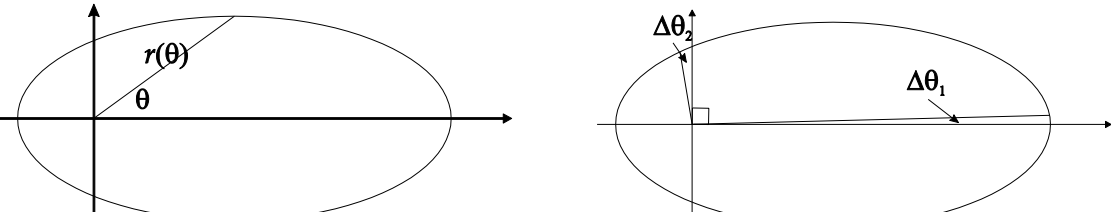
4. Let us assume that an observer using a hypothetical, far-infrared, Earth-sized telescope (wavelength range 20 to 640 μm) found a static and neutral supermassive black hole with a mass of $2.1 \times 10^{10} M_{\odot}$. Determine the maximum distance at which this black hole can be resolved by the observer.

Answer and Marking Scheme:

1	<p>Identify the relationship between the telescope's resolution and the angular size of black hole</p> <p>This problem can be solved by using relationship between the telescope's resolution and the angular size of black holes which can be seen in this equation:</p> $\phi_{telescope} \leq \phi_{blackhole} \Rightarrow 1.2 \frac{\lambda}{D_{telescope}} \leq \frac{\text{Diameter of event horizon}}{\text{Observer Distance to galactic's the center}}$ $1.2 \frac{\lambda}{D_{telescope}} \leq \frac{2R_{blackhole}}{d}$	25
2	<p>Identify the relation for determining the distance</p> <p>Then, it is assumed that the radius of black hole is the Schwarzschild radius; and the telescope is also Earth-sized telescope. Therefore:</p> $1.2 \frac{\lambda}{2R_{Earth}} \leq \frac{2R_{Schwarzschild,BH}}{d}$ $d \leq \frac{4 \times R_{Schwarzschild,BH} \times R_{Earth}}{1.2 \times \lambda} \Rightarrow d \leq \frac{4 \times M_{Black\ Hole} \times R_{Schwarzschild,Sun} \times R_{Earth}}{1.2 \times M_{Sun} \times \lambda}$ $d \leq \frac{4 \times 2.1 \times 10^{10} M_{Sun} \times R_{Schwarzschild,Sun} \times R_{Earth}}{1.2 \times M_{Sun} \times \lambda} \Rightarrow d \leq \frac{8.4 \times 10^{10} \times R_{Schwarzschild,Sun} \times R_{Earth}}{1.2 \times \lambda}$	50
3	<p>Calculating the maximum-observer distance</p> <p>To find the maximum distance, then we should use the minimum λ. Therefore:</p> $d_{max} = \frac{8.4 \times 10^{10} \times R_{Schwarzschild,Sun} \times R_{Earth}}{1.2 \times \lambda_{min}} = \frac{8.4 \times 10^{10} \times 2.95 \times 10^3 \times 6.3708 \times 10^6}{1.2 \times 20 \times 10^{-6}} m = 6.6 \times 10^{25} m$	25

5. An observer is trying to determine an approximate value of the orbital eccentricity of a man-made satellite. When the satellite was at apogee, it was observed to have moved by $\Delta\theta_1 = 2'44''$ in a short time. When the radius vector connecting Earth and the satellite is perpendicular to the major axis (true anomaly is equal to 90°), within the same duration of time, it was observed to have moved by $\Delta\theta_2 = 21'17''$. Assume that the observer is located at the center of the Earth. Find an approximate value of the eccentricity of the satellite's orbit.

Answer and Marking Scheme:

1	<p>Identify expression for $r(\theta)$.</p> <p>We know that $r(\theta) = \frac{a(1-e^2)}{1-e\cos\theta}$. Hence</p> $r(0) = \frac{a(1-e^2)}{1-e} = a(1+e)$ $r\left(\frac{\pi}{2}\right) = \frac{a(1-e^2)}{1-e \times 0} = a(1-e^2)$ 	30
2	<p>Estimate area of sectors using area of triangles.</p> <p>Estimate the area of swept out sectors as the area of sectors of circles</p> $A(S_1) \approx 0.5 \times \Delta\theta_1 \times (r(0))^2 = \Delta\theta_1 (a(1+e))^2$ $A(S_2) \approx 0.5 \times \Delta\theta_2 \times \left(r\left(\frac{\pi}{2}\right)\right)^2 = \Delta\theta_2 (a(1-e^2))^2$	20
3	<p>Use Kepler's second law to find a relation between the ratio $\frac{\Delta\theta_1}{\Delta\theta_2}$ and the eccentricity e.</p> <p>Use Kepler's second law to obtain that</p> $A(S_1) = A(S_2)$ $\Delta\theta_1 \times (a(1+e))^2 = \Delta\theta_2 \times (a(1-e^2))^2$ <p>Thus,</p> $\frac{\Delta\theta_1}{\Delta\theta_2} = \left(\frac{1-e^2}{1+e}\right)^2 = (1-e)^2 = \frac{2'44''}{21'17''} \approx 0.12843$	30
4	<p>Obtain an estimate value of the eccentricity.</p> <p>Thus, the eccentricity is $e \approx 1 - \sqrt{0.12843} = 0.64$</p>	20

6. At the start of every observation, a radio telescope is pointed at a point-source calibrator that has a known flux density of 21.86 Jy outside the Earth's atmosphere. However, on a certain date, the measured flux density of the calibrator source was 14.27 Jy. If the calibrator source was at an altitude of 35 degrees, estimate the zenith atmospheric optical depth, τ_z .

Answer and Marking Scheme:

1	<p>Atmospheric optical depth τ_A can be determined from the measured flux, that is:</p> $S_{\text{meas}} = \exp(-\tau_A) S_{\text{real}}$ $14.27 = \exp(-\tau_A) \times 21.86 \rightarrow \exp(-\tau_A) = 14.27/21.86 = 0.65$ <p>Then</p> $\tau_A = -\ln(0.65) = 0.43$	50
2	<p>Now we have:</p> $\tau_A = \tau_z \sec z$ $\tau_z = \tau_A \cos z$ $= 0.43 \times \cos(90 - 35)^\circ = 0.25$ <p>where z is zenith angle.</p>	50

7. A galaxy at the boundary of a galaxy cluster of radius 10 Mpc is expected to escape from the cluster if it has an initial velocity of at least 700 km/s relative to the center of the cluster. Calculate the density of the cluster.

Answer and Marking Scheme:

1	We have the total energy of the escaping galaxy: $E_T = E_K + E_P = \frac{mV^2}{2} - \frac{GMm}{R}$	15
2	Where V is velocity of galaxy and R radius of galaxy cluster Galaxy will escape from galaxy cluster if the total energy, $E_T = 0$	10
3	$E_T = 0 \rightarrow \frac{mV^2}{2} - \frac{GMm}{R} = 0$ $V^2 = \frac{2GM}{R}$	15
4	Escape velocity $V^2 = \frac{2G \left(\frac{4\pi}{3} R^3 \rho \right)}{R}$	20
5	Density of cluster $\rho = \frac{3V^2}{8\pi GR^2}$	15
6	$V=700$ km/s $R=10$ Mpc $G=6.6378 \times 10^{-11}$ Nm ² kg $\rho=9.20121 \times 10^{-27}$ kg/m ³	25

9. Assume that the Sun is a perfect blackbody. Venus is also assumed to be a blackbody, with temperature T_V , and it is in thermal equilibrium (i.e. it is radiating about as much energy as it receives from the Sun) at its orbital distance of 0.72 AU. Suppose that at closest approach to Earth, Venus has an angular diameter of about 66 arcsec. What is the flux density of Venus at the closest approach to Earth as observed by a radio telescope at an observing frequency of 5 GHz?

[Add the formula on the Table of Constants (also mentioning that the Rayleigh-Jeans formula is a derivation from Planck's formula)]

Answer and Marking Scheme:

1	Solar luminosity is $L_{\odot} = 4\pi R_{\odot}^2 \sigma T_{\odot}^4$ $= 4 \times 3.14 \times (6.96 \times 10^{10})^2 (5.67 \times 10^{-5})(5780)^4 \text{ ergs/sec}$ $= 3.9 \times 10^{33} \text{ ergs/sec}$	10
2	Flux of the Sun intercepted by Venus is $L_V = (A_{V(\text{projected})}/A_{(\text{sphere}, 0.72\text{au})}) \times L_{\odot}$ Also, $L_V = 4\pi R_V^2 \sigma T_V^4$ So, $4\pi R_V^2 \sigma T_V^4 = (\pi R_V^2/4\pi d^2) \times L_{\odot} = (\pi R_V^2/4\pi d^2) \times 4\pi R_{\odot}^2 \sigma T_{\odot}^4$	30
3	Therefore, $T_V^4 = (R_{\odot}^2/4d^2) \times T_{\odot}^4 = [(6.957 \times 10^5 \text{ km})^2/4 \times (0.72 \times 149.6 \times 10^6)^2] \times (5780)^4$ $= (483998490000/46407499776000000) \times (5780)^4 = (4.84 \times 10^{11}/4.64 \times 10^{16}) \times (5780)^4$ $= 1.0429316216908190111241385226915e-5 \times 1114577187760656$ $= 1.04 \times 10^{-5} \times 1.11 \times 10^{15}$ $= 11624277939.308134327737156481455 = 1.1624 \times 10^{10}$ $T_V = 328.5 \text{ K}$	20
4	The flux density of Venus at closest approach to Earth at an observing frequency of 5 GHz: ($1'' = 4.84 \times 10^{-6} \text{ rad}$) $S_{5\text{GHz}} = B_{5\text{GHz}} \Omega = (2kT_b v^2/c^2) \Omega \quad (\rightarrow T_b = T_V)$ $= (2 \times 1.38 \times 10^{-16} \times 328.5 \times (5 \times 10^9)^2 \times \pi (66 \times 4.84 \times 10^{-6})^2) / (3 \times 10^{10})^2$ $= 8.07 \times 10^{-22} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$ $= 80.7 \text{ Jy}$	40

10. A molecular hydrogen cloud is known to have a temperature $T = 115$ K. If the atoms (assumed spherical) in this cloud have masses of about 5×10^{-23} g and radii of 3.5 \AA , and the separation distance between the hydrogen atoms is [Placeholder]. Assume that the molecules are homogenous and in thermal equilibrium. Estimate the frequency at which they will radiate due to molecular rotational excitation.

[Add the separation distance between hydrogen atom.]

Answer and Marking Scheme:

1	<p>The molecules rotate with angular velocity ω and for spherical molecules, their moment of inertia is $I = \frac{2}{5} mr^2$, i.e., [The formula of the moment of inertia will be corrected accordingly]</p> $I = \frac{2}{5} mr^2 = \frac{2}{5} (5 \times 10^{-23} \text{ g}) (3.5 \times 10^{-8} \text{ cm})^2 = 2.45 \times 10^{-38} \text{ g cm}^2$	30
2	<p>In thermal equilibrium, we have: [Change formula into: kT]</p> $\frac{1}{2} I \omega^2 = \frac{3}{2} kT$	30
3	<p>So we can estimate:</p> $\omega = (3kT/I)^{1/2} = (3 \times 1.38 \times 10^{-16} \text{ J} / 2.45 \times 10^{-38} \text{ g cm}^2)^{1/2} = 1.39 \times 10^{12} \text{ s}^{-1}$ $\nu = \omega/2\pi = 2.2 \times 10^{11} \text{ Hz}$	40

11.	The mass density of an object is inversely proportional to the radial distance from the center of the object with a factor of proportionality $\alpha = 5.0 \times 10^{13} \text{ kg/m}^2$. If the escape velocity at the surface of the object is $v_0 = 1.5 \times 10^4 \text{ m/s}$, calculate the total mass of the object.	
	Answer	
1	$\frac{1}{2}mv_0^2 - \frac{GmM}{R} = 0 \rightarrow R = \frac{2GM}{v_0^2}$	20
2	$\Delta m(r) = \rho 4\pi r^2 \Delta r = 4\pi\alpha r \Delta r \rightarrow M = 2\pi\alpha R^2$ $(dm(r) = \rho 4\pi r^2 dr = 4\pi\alpha r dr \rightarrow M = 2\pi\alpha R^2)$	30
3	$M = \alpha 2\pi R^2 = \alpha 2\pi \frac{4G^2 M^2}{v_0^4} \rightarrow M = \frac{v_0^4}{8\pi\alpha G^2}$	30
4	$M = \frac{v_0^4}{8\pi\alpha G^2} = \frac{(1.5 \times 10^4 \text{ ms}^{-1})^4}{8\pi(5 \times 10^{13} \text{ kgm}^{-2})(6.67 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2})^2} = 9.1 \times 10^{21} \text{ kg}$	20



12.	A proton with a kinetic energy of 1GeV propagates out from the surface of the Sun towards the Earth. Neglecting the magnetic field of the Sun, calculate the travel time of the proton as seen from the Earth.	
	Answer	
1	$E_0 = m_0c^2 = 941\text{ MeV}$	20
2	$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{K}{E_0} + 1 = \frac{1000\text{ MeV}}{941\text{ MeV}} + 1 = 2.062$	20
3	$1 - \frac{v^2}{c^2} \approx \frac{1}{4} \rightarrow v \approx 0.87c$	30
4	$\text{Elapse time} = \frac{\text{Earth - Sun distance}}{\text{speed}} = \frac{1.50 \times 10^{11}\text{ m}}{0.87 \times (3 \times 10^8)\text{ m/s}}$ $= 9.6\text{ minutes}$	30



13. Two protons collide to produce deuteron, positron, and neutrino in the center of a star. Write down the reaction equation and calculate the minimum temperature needed for this reaction. Assume that the neutrino mass is negligible.

	Answer	
1	$p + p + \Delta E \rightarrow D + e^+ + \nu_e$	20
2	$\Delta E = (m_e + m_D - 2m_p)c^2 = (m_e + (m_p + m_n) - 2m_p)c^2$ $= (m_e + m_n - m_p)c^2$	20
3	ΔE $= (9.109 \times 10^{-31}$ $+ 1.675 \times 10^{-27} - 1.673 \times 10^{-27}) \text{kg} (2.99792 \times 10^8 \text{ms}^{-1})^2$ $= 2.6198 \times 10^{-13} \text{ J}$	20
4	$\Delta E \sim k_B T \rightarrow T \sim \frac{\Delta E}{k_B}$	20
5	$T = \frac{2.6198 \times 10^{-13} \text{ J}}{8.315 \frac{\text{J}}{\text{mol K}}} = \frac{2.6198 \times 10^{-13} \text{ mol K}}{8.315} \times 6.02 \times 10^{23} \frac{1}{\text{mol}}$ $= 1.894 \times 10^{10} \text{ K}$	20

14. Suppose we live in a static and infinitely large universe where the average density of stars is $n = 10^9 \text{ Mpc}^{-3}$ and the average stellar radius is equal to the solar radius. Assume that standard Euclidean geometry holds true in this universe. How far, on average, could you see in any direction before your line of sight strikes a star? Please write your answer in Mpc.

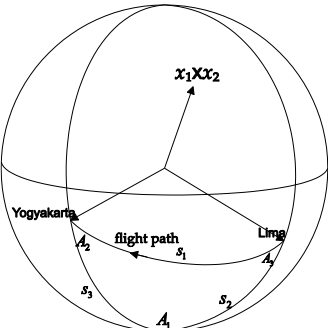
Answer and Marking Scheme:

1.	<p>To decide how far one can see on average in a universe filled with spherical objects of radius R, it is simplest to think of a long cylinder along the line of sight. If an object is closer than R to the line of sight, then the line of sight intersects its surface. For a distance ℓ, the cylindrical volume which would contain such objects is $\pi R^2 \ell$. If the density of objects is n, then the volume that will on average contain one object is defined by $\pi R^2 \ell n = 1$ and the average distance to which we see before our vision is blocked is</p> $\ell = \frac{1}{\pi R^2 n}$	50
2.	<p>The average radius R in Mpc</p> $R = \frac{6.96 \times 10^8 \text{ m}}{3.086 \times 10^{22} \text{ m Mpc}^{-1}} = 2.3 \times 10^{-14} \text{ Mpc}$	25
3.	<p>Then</p> $\ell = \frac{1}{3.14 (2.3 \times 10^{-14} \text{ Mpc})^2 (10^9 \text{ Mpc}^{-3})} = 6.3 \times 10^{17} \text{ Mpc}$	25

15. An airplane was flying from Lima, capital of Peru ($12^{\circ}2'S$ and $77^{\circ}1'W$) to Yogyakarta ($7^{\circ}47'S$ and $110^{\circ}26'E$), near the venue of the 9th IOAA. The airplane chooses the shortest flight path from Lima to Yogyakarta. Find the latitude of the southernmost point of the flight path.

Answer and Marking Scheme:

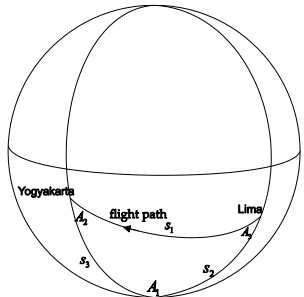
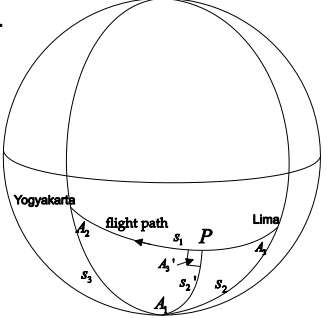
1st version

1	<p>Determine coordinates of Lima and Yogyakarta. Assume that the equator is the $z = 0$ plane and the Earth is a unit sphere. The Cartesian coordinates of the Lima is $\begin{aligned} \mathbf{x}_1 &= (x_1, y_1, z_1) \\ &= (\cos(-12^{\circ}2') \cos(282^{\circ}58'48), \cos(-12^{\circ}2') \sin(282^{\circ}58'48), \sin(-12^{\circ}2')) \\ &= (0.2197308726, -0.9530236794, -0.2084807188) \end{aligned}$ The Cartesian coordinates of the Yogyakarta is $\begin{aligned} \mathbf{x}_2 &= (x_2, y_2, z_2) = (\cos(-7^{\circ}47') \cos(110^{\circ}26'), \cos(-7^{\circ}47') \sin(110^{\circ}26'), \sin(-7^{\circ}47')) \\ &= (-0.3459009563, 0.9284459898, -0.1354273699) \end{aligned}$</p>	15
2	<p>Determine the normal of the plane defined by Lima, Yogyakarta, and center of the Earth. The normal of the plane containing the two places and the center of the Earth is $\mathbf{x}_1 \times \mathbf{x}_2 = (a, b, c) = (0.3226285776, 0.1018712545, -0.12564355210)$</p>	20
3	<p>Find the equation of the (shortest flight) path from Lima to Yogyakarta. Let the equation of the plane be $ax + by + cz = 0$ or equivalently $y = -a'x - c'z$ where $a' = -3.16702272082062$ and $c' = 1.23335628599724$. For the sake of simplicity, from now let us denote a' and c' as a and c, respectively. Intersection of the Earth and the plane is the large circle $x^2 + (-ax - cz)^2 + z^2 = 1$ or equivalently It is also the equation for the shortest flight path from Lima to Yogyakarta.</p> 	25
4	<p>Determine the z-value of the southernmost point of the path (at the point, the discriminant of the equation equals zero). $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$</p> <p>The plane $z = m$, it intersects the large circle at at either at one or two points. The number of intersection point at the highest or at the lowest point is one, i.e. when the discriminant is zero.</p> $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$ <p>The discriminant is</p>	30

	$D = (2acm)^2 - 4(1 + a^2)[(1 + c^2)m^2 - 1]$ $= 4 + 4a^2 - (4 + 4a^2 + 4c^2)m^2$ <p>Therefore $D = 0$, if $m = \pm \sqrt{\frac{1+a^2}{1+a^2+c^2}} = 0.937444937497383$. Then choose $z = -0.937444937497383$</p>	
5	<p>Determine the latitude of the southernmost point. Thus, the latitude of the point is $\sin^{-1} z = 1.21521692653748 = 69.6268010651290^\circ = 69^\circ 37' 36'' S$.</p>	10

2nd version (using the Law of Cosines for sides)

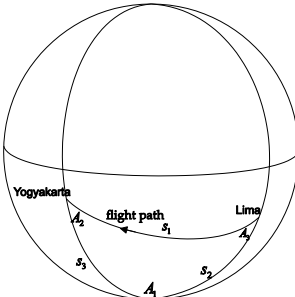
Answer and Marking Scheme:

1	<p>Determine length of the sides of the triangle $s_2 = 90^\circ - 12^\circ 2' = 77^\circ 58' = 1.3608 \text{ rad}$, $s_3 = 90^\circ - 7^\circ 47' = 82^\circ 13' = 1.4350 \text{ rad}$; $A_1 = (180^\circ - 77^\circ 1') + (180^\circ - 110^\circ 26') = 172^\circ 33' = 3.0116 \text{ rad}$</p> <p>Using Law of Cosines for sides to obtain $s_1 = 2.7724 \text{ rad}$</p>		20
2	<p>Determine the angles of the triangle Then use $\cos s_2 = (\cos s_1)(\cos s_3) + (\sin s_1)(\sin s_3)(\cos A_2)$ to obtain $A_2 = 0.35921 \text{ rad}$. Similarly, use Law of Cosines $\cos s_3 = (\cos s_1)(\cos s_2) + (\sin s_1)(\sin s_2)(\cos A_3)$ to get that $A_3 = 0.3642 \text{ rad}$.</p>	30	
3	<p>Use Law of Cosines for sides to construct an equation in $\cos s_2'$. If P is the southernmost point, then the angle A_3 is a right angle, $A_3 = \frac{\pi}{2}$, the Law of Thus, Then substitute it to get</p>		20
4	<p>Solve the equation Let $x = \cos s_2'$. Then</p>	20	

	$x = \frac{0.018328}{x} + \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ <p>Multiply both sides by x to get</p> $x^2 = 0.018328 + x \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ $x^2 - 0.018328 = x \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ $(x^2 - 0.018328)^2 = (x^2 - 0.018328) \times (0.92756)^2$ $x^4 - 0.89702x^2 + 0.016105 = 0$ <p>which is a quadratic equation of x^2. The roots are ± 0.13538 and ± 0.93739.</p>	
5	<p>Determine the Latitude</p> <p>Thus,</p> $s'_2 = \cos^{-1} 0.93739 = 0.35574 \text{ rad} = 20^\circ 23'$ <p>or</p> $s'_2 = \cos^{-1} 0.13538 = 1.4350 \text{ rad} = 82^\circ 13'$ <p>The latitude is $69^\circ 37'S$ or $7^\circ 47'S$. The second one is impossible because it higher then the latitude of Lima. Thus the answer is</p> $69^\circ 37'S$	10

3rd version (using Napier's rule)

Answer and Marking Scheme:

1	<p>Determine length of the sides of the triangle</p> $s_2 = 90^\circ - 12^\circ 2' = 77^\circ 58' = 1.3608 \text{ rad}, s_3 = 90^\circ - 7^\circ 47' = 82^\circ 13' = 1.4350 \text{ rad}; A_1 = (180^\circ - 77^\circ 1') + (180^\circ - 110^\circ 26') = 172^\circ 33' = 3.0116 \text{ rad}$ <p>Using Law of Cosines for sides to obtain $s_1 = 2.7724 \text{ rad}$</p>		20
2	<p>Determine the angles of the triangle</p> <p>Then use</p> $\cos s_2 = (\cos s_1)(\cos s_3) + (\sin s_1)(\sin s_3)(\cos A_2)$ <p>to obtain $A_2 = 0.35921 \text{ rad}$. Similarly, use Law of Cosines</p> $\cos s_3 = (\cos s_1)(\cos s_2) + (\sin s_1)(\sin s_2)(\cos A_3)$ <p>to get that $A_3 = 0.3642 \text{ rad}$.</p>	30	

3	<p>Find the latitude of the point</p> <p>Let P be the southernmost point on the flight path. Since the triangle has a right triangle at P, then use one of Napier's rules to get $s_2' = 0.35576 \text{ rad} = 20^\circ 23'$. Therefore, the latitude of P is</p>	50
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