Theoretical Exam - Short Questions

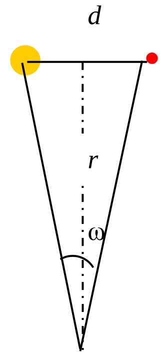
1. What would be the temperature on the Earth's surface if we ignore the greenhouse effect but take into account its non-vanishing albedo? Assume that the Earth's orbit around the Sun is circular.

Answer : $T_{\oplus} = T_{\odot}\,(1-\alpha)^{1/4}\sqrt{\dfrac{R_{\odot}}{2r_{\oplus}}} = 255\text{ K}$ or $\mathbf{T_{\oplus} = -19^{o}\,C}$

(Note: $r_{\oplus}$ = average Earth-Sun distance, the albedo, $a = 0.39$ is given in the *additional material*).

2. Let us assume that we observe a Jupiter-like planet orbiting around a star at an average distance $d = 5$ AU. It has been found that the distance of this system from us is $r = 250$ pc. What is the minimum diameter, D, that a telescope should have to be able to resolve the two objects (star and planet)? We assume that the observation is done in the optical part of the electromagnetic spectrum (λ~500nm), outside the Earth's atmosphere and that the telescope optics are perfect (diffraction limited).

Answer: From the figure we have:



$$\omega(rad) \approx \frac{d}{r} = \frac{5AU}{250\,pc} = \frac{5\times1.5\times10^{11}m}{250\times3.09\times10^{16}m} = 1.03\times10^{-7}rad$$

Let us assume that D is the minimum diameter of our space telescope.

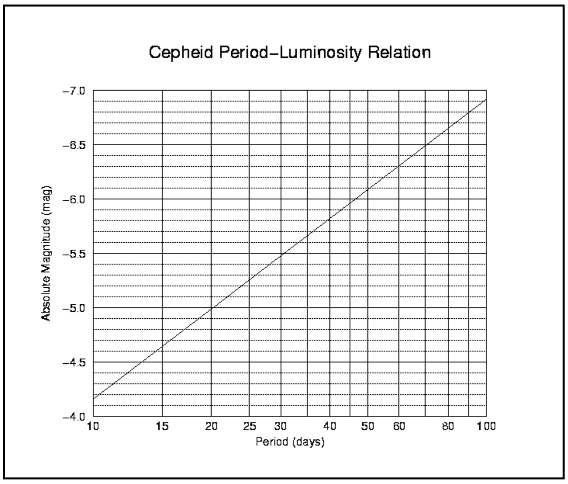
Its angular resolution is

$$\Delta\theta = 1.22\times\frac{\lambda}{D} \rightarrow 1.03\times10^{-7}rad = 1.22\times\frac{500\times10^{-9}m}{D} \rightarrow \boldsymbol{D = 6\ m}$$

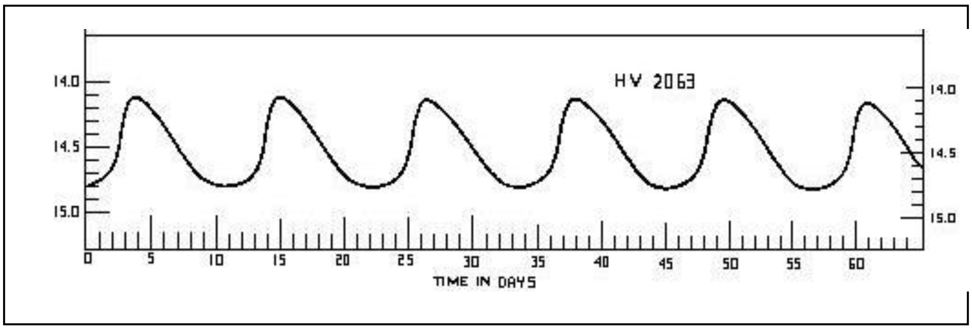
3. It is estimated that the Sun will have spent a total of about t_1 = 10 billion years on the main sequence before evolving away from it. Estimate the corresponding amount of time, t_2, if the Sun were 5 times more massive.

Answer: For the average luminosity of a main sequence star we have: $L \propto M^4$ (where M the initial mass of the star). We assume that the total energy E that the star produces is proportional to its mass $E \propto M$ **.** Therefore the amount of time that the star spends on the main sequence is approximately $t_{MS} \approx \dfrac{E}{L} \propto \dfrac{M}{M^4} \approx M^{-3}$. Therefore, $\dfrac{t_1}{t_2} \approx \dfrac{1^{-3}}{5^{-3}} = 5^3 \rightarrow t_2 = \dfrac{1}{125}\times10^{10}yr$ or $\mathbf{t_2 = 8\times10^7\ yr}$.

4. Figure 1 shows the relation between absolute magnitude and period for classical cepheids. Figure 2 shows the light curve (apparent magnitude versus time in days) of a classical cepheid in the Small Magellanic Cloud. (a) Using these two figures estimate the distance of the cepheid from us. (b) Revise your estimate assuming that the interstellar extinction towards the cepheid is $A = 0.25$ mag.



m



Answer : (a) From Figure 2, the period of the cepheid is P ~ 11 days and its average apparent magnitude is ~(14.8+14.1)/2 mag, i.e. m = 14.45 mag. **(1 point)**

[A careful student will notice that the graph is not *upside-down* symmetrical so he/she choose some value closer to the bottom; that is m=14.5 mag. **(3 point)]**

From Figure 1, we derive that for a period of 11 days the expected absolute magnitude of the cepheid is $M \approx -4.2$. **(1 point)**

[A careful student will notice that the graph is logarithmic so he/she choose a value closer to 4.3] **(3 point)]**

Using the formula $m - M = -5 + 5\log r$, where r is the distance of the cepheid, we get

$\log r = (14.45 + 4.2 + 5)/5 = 4.73$, thence $r = 10^{4.73} \approx 57500$ pc or ***r* = 57.5 kpc** **(3 points)**

(b) Assuming A = 0.25, then $\log r = (14.45+4.2+5+0.25)/5 = 4.78$, thence $r = 10^{4.78}$ ~53000 pc or

***r* = 64.5 kpc.** **(1 point)**

5. The optical spectrum of a galaxy, whose distance has been measured to be 41.67 Mpc, shows the Balmer Ha line ($\lambda_o = 656.3$ nm) redshifted to $\lambda = 662.9$ nm. (a) Use this to calculate the value of the Hubble constant, H_o. (b) Using your results, estimate the age of the Universe.

Answer: (a) $z = \dfrac{\lambda - \lambda_o}{\lambda_o} = \dfrac{662.9 - 656.3}{656.3} \approx 0.01$. This is small enough that we can use the classical equation for the expansion of the Universe. $H_o = \dfrac{cz}{r} = \dfrac{3\times10^5\, km\, s^{-1} \times 10^{-2}}{41.67\, Mpc}$ or $\boldsymbol{H_o} = 72.0 \dfrac{km\, s^{-1}}{Mpc}$

(b) $t_H \approx \dfrac{1}{H_o}$ or $\boldsymbol{t_H}$ **= 13.9 Gyr**

6. A star has an effective temperature $T_{eff} = 8700$ K, absolute magnitude $M = 1.6$ mag and apparent magnitude $m = 7.2$ mag. Find (a) the star's distance, r, (b) its luminosity, L, and (c) its radius, R. (Ignore absorption).

Answer : (a) Its distance is calculated from equation: $m - M = 5 \log (r) - 5$,
or $7.2 - 1.6 + 5 = 5\log(r) \rightarrow \log(r) = 2.12$ and $\boldsymbol{r} =$ **132 pc**

(b) Its Luminosity is calculated from equation: $M_\odot - M = 2.5 \log\left(\dfrac{L}{L_\odot}\right)$ or

$4.8 - 1.6 = 2.5 \log\left(\dfrac{L}{L_\odot}\right)$ or $\log\left(\dfrac{L}{L_\odot}\right) = 1.28 \rightarrow \left(\dfrac{L}{L_\odot}\right) = 19.05$ or $L = 19.15\times L_\odot =$ $19.15\times3.9\times10^{33}$ and **L = 7.4×10^{34} erg sec^{-1}**

(c) Its radius can be easily calculated from equation: $L = 4\pi\sigma R^2 T^4_{eff}$, from which

$R = \dfrac{1}{T^2_{eff}}\sqrt{\dfrac{L}{4\pi\sigma}}$ from which we get: $\boldsymbol{R} =$ **1.35×10^{11} cm**

7. A star has visual apparent magnitude $m_V = 12.2$ mag, parallax $\pi = 0''.001$ and effective temperature $T_{eff} = 4000$ K. Its bolometric correction is $B.C. = -0.6$ mag. (a) Find its luminosity as a function of the solar luminosity. (b) What type of star is it? (i) a *red giant*? (ii) a *blue giant*? or (iii) a *red dwarf*? Please write (i), (ii) or (iii) in your answer sheet.

Answer : (a) First its bolometric magnitude is calculated from equation: $M_V - m_V = 5 - 5 \log (r)$
or equivalent: $M_V - m_V = 5 + 5 \log \pi \rightarrow M_V = 12.2 + 5 + 5\log(0''.001) = 12.2 + 5 - 15 = 2.2$ mag. Its barycentric correction is: $B.C. = M_{bol} - M_V$ and $M_{bol} = B.C. + M_V$ or $M_{bol} = -0.6 + 2.2$ or $M_{bol} = 1.6$ mag . Then its Luminosity is calculated from:

$M_\odot - M_{bol} = 2.5 \log\left(\dfrac{L}{L_\odot}\right)$, or $4.8 - 1.6 = 2.5 \log\left(\dfrac{L}{L_\odot}\right)$ or $\log\left(\dfrac{L}{L_\odot}\right) = 1.28$ and $\boldsymbol{L} =$ **19.1** $\boldsymbol{L_\odot}$

(b) Type of star: A star with $M_{bol} = 1.6$ mag, $L = 19.1\, L_\odot$ and $T_{eff} = 4000$ K is much brighter and much cooler than the Sun (see *Table of constants*). Therefore it is **(i)** a red giant star.

8. A binary system of stars consists of star (*a*) and star (*b*) with brightness ratio 2. The binary system is difficult to resolve and is observed from the Earth as <u>one star</u> of 5th magnitude. Find the apparent magnitude of each of the two stars (m_a, m_b).

Answer : The apparent magnitude of star (*a*) is m_a, of star (b) is m_b and that of the system as a whole is m_{a+b}. The corresponding apparent brightnesses are ℓ_a, ℓ_b and $\ell_{a+b} = \ell_a + \ell_b$.
For star (*a*) :

$m_{a+b} - m_a = -2.5\log\left(\frac{\ell_a+\ell_b}{\ell_a}\right)$ and because $\frac{\ell_b}{\ell_a}=\frac{1}{2}$, we get $m_a = m_{a+b} + 2.5\log(1+½)$ or

$m_a = 5 + 2.5\log(3/2)$ and finally $\boldsymbol{m_a}$ **= 5.44 mag** . Similarly for star (*b*):

$m_{a+b} - m_b = -2.5\log\left(\frac{\ell_a+\ell_b}{\ell_b}\right)$ and because $\frac{\ell_a}{\ell_b}=2$, we get $m_b = m_{a+b} + 2.5\log(3)$ or

$m_b = 5 + 2.5\log(3)$ and finally $\boldsymbol{m_b}$ **= 6.19 mag**.

9. Find the equatorial coordinates (*hour angle* and *declination*) of a star at Madrid, geographic latitude φ = 40°, when the star has zenith angle z = 30° and azimuth A = 50°

Answer : From the position triangle $\Pi Z_v \Sigma$ (Figure 3) of the star, Σ, we get, by using the *cosine law* for a spherical triangle:

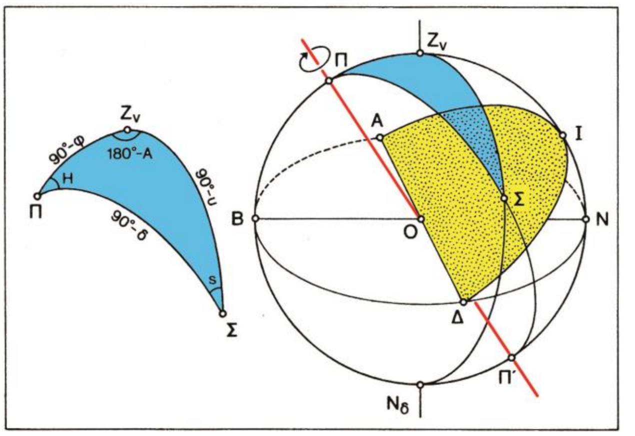


Figure 3. The position triangle

cos (90 – δ) = cos(90 – *a*)×cos(90 – φ) + sin (90 – *a*)×sin(90 – φ)×cos (180 – A)

where δ is the star's *declination*, *a* its *altitude* (*a* = 90° – z), φ the *geographical latitude* of the observer, H the stars *hour angle* and A the star's *azimuth*.
This can be written as:

sinδ = cos z×sin φ – sin z×cosφ×cos A or
sinδ = cos30°×sin40° – sin30°×cos40°×cos50°
or
sin δ =0.866×0.643 – 0.500×0.766×0.643 = 0.311. **δ = 18° 07′**

Using the sine law for the spherical triangle, we get:

$$\frac{\sin H}{\sin(90-a)} = \frac{\sin(180^o - A)}{\sin(90^o-\delta)} \text{ or } \frac{\sin H}{\sin z} = \frac{\sin A}{\cos\delta} \rightarrow \sin H = \sin 50^o \times \frac{\sin 30^o}{\cos(18^o 07')} = \frac{0.766\times 0.5}{0.950} \text{ or}$$

sin H = 0.403. Therefore: H = 23° 46′ or **H = 1^h 35^m 04^s**.

10. In the centre of our Galaxy, in the intense radio source Sgr A*, there is a black hole with large mass. A team of astronomers measured the angular distance of a star from Sgr A* and its orbital period around it. The angular distance was 0.12″ (arcsec) and the period was 15 years. ~~The distance of our solar system from the centre of the Galaxy is 8 kpc~~[Put it in Constants]. Calculate the mass of the black hole in solar masses.

Answer: $F = -\frac{GM_{BH}M_*}{R^2} = -\frac{M_* v^2}{R}$

But $v = \frac{2\pi R}{P}$. Therefore $\frac{GM_{BH}}{R^2} = 4\pi^2 \frac{R}{P^2}$ or $GM_{BH} = 4\pi^2 \frac{R^3}{P^2}$

Similarly:

$GM_{\odot} = 4\pi^2 \frac{(1AU)^3}{(1yr)^2}$ ή $G = \frac{1}{M_{\odot}} 4\pi^2 \frac{(1AU)^3}{(1yr)^2}$

From Kepler's 3rd law we get: $\frac{M_{BH}}{M_{\odot}} = \frac{(R/1AU)^3}{(P/1yr)^2}$

Inserting the given data we find the distance of the star from the black hole:

$$R = \frac{0.12}{200,000}(8000)(3\times10^{18}cm) = 1.4\times10^{16}cm = 900AU$$

Therefore: $\frac{M_{BH}}{M_{\odot}} = \frac{(900)^3}{(15)^2} = 3\times10^6$ form which we calculate the mass of the black hole:

$$M_{BH} = 3\times10^6 M_{\odot}$$

11. What is the maximum altitude, a_M(max), that the Full Moon can be observed from Thessaloniki? The geographical latitude of Thessaloniki is $\varphi_\Theta = 40°37'$.

Answer: In order to have Full Moon, the Moon should be diametrically opposite the Sun, i.e. the three bodies, Sun – Earth – Moon should be on a straight line. If the orbital plane of the Moon coincided with the ecliptic, the maximum altitude of the Full Moon would be $90° – \varphi_\Theta + 23.5°$. Because the orbital plane of the Moon is inclined by 5.14° (5°18′) to the plane of the ecliptic, the maximum angle is larger: 90° – 40°37′ + 23.5° + 5°18′, or

$$\boldsymbol{\alpha_M(\max) = 79°49'}$$

12. *Sirius A*, with visual magnitude $m_V = –1.47$ (the brighter star on the sky) and with stellar radius $R_A = 1.7R_\odot$, is the primary star of a binary system. The existence of its companion, *Sirius B*, was deduced from astrometry in 1844 by the well known mathematician and astronomer Friedrich Bessel, before it was directly observed. Assuming that both stars were of the same spectral type and that *Sirius B* is fainter by 10 mags ($\Delta m = 10$), calculate the radius of *Sirius B*.

Answer: The distance of the two stars from our solar system is the same. Therefore

$$m_B - m_A = 2.5 \log \frac{\frac{L_A}{4\pi r^2}}{\frac{L_B}{4\pi r^2}} = 2.5 \log \frac{L_a}{L_B}$$

From which we get $L_A = 10^4 L_B$. From equation $L = 4\pi R^2\sigma T_{eff}^4$ βwe get

$$L_A/L_B = (R_A/R_B)^2 (T_A/T_B)^4$$

Assuming that the two stars belong to the same spectral type (and therefore $T_A = T_B$) we get

$$R_B = 0.01R_A = 0.01\times1.7\times696000 \text{ km or } \boldsymbol{R_B = 11832 \text{ km}}$$

13. Recently in London, because of a very thick layer of fog, the visual magnitude of the Sun, became equal to the (usual – as observed during cloudless nights) magnitude of the full Moon. Assuming that the reduction of the intensity of light due to the fog is given by an exponential equation, calculate the exponential coefficient, τ, which is usually called *optical depth*.

Answer: The absorption due to the fog in London is obviously
$A = –26.8 – (–12.5) = –14.3$ mag.

Rearranging the equation $I_\nu(r) = I_\nu(0) \times e^{-\tau}$, we get $\frac{I_\nu(0)}{I_\nu(r)} = e^{\tau}$, or

$A = \Delta m = 2.5 \times \log(e) \times (-\tau)$ or $\tau = (–A)/(2.5 \times \log e) = 14.3/1.08$ ή $\boldsymbol{\tau = 13.2}$

14. What is the hour angle, H, and the zenith angle, z, of *Vega* in Thessaloniki ($\lambda_1 = 1^h32^m$, $b_1 = 40°37^m$), at the moment it culminates at the local meridian of Lisbon ($\lambda_2 = +00^h36^m$, $b_2 = +39°43'$)?

Answer: By definition at the moment when the star culminates in Lisbon, its hour angle is exactly 0°. Therefore its hour angle in Thessaloniki is 0° + ($\lambda_1 – \lambda_2$) or $\boldsymbol{H = 00^h56^m}$.

The zenith distance at Lisbon ($90° – b_1 + d$)

Using the cosine law equation $\cos z = \cos(90-\varphi)\times\cos(90-d) + \sin(90-\varphi)\times\sin(90-d) \times\cos H$, the zenith distance at Thessaloniki can be calculated.

15. The Doppler shift of three remote galaxies has been measured with the help of Spectral observations:

Galaxy	Redshift, z
3C 279	0.536
3C 245	1.029
4C41.17	3.8

(a) Calculate their recession velocity. (b) At what percentage of the speed of light they recede? (c) What is the present distance of each galaxy?

Answer: The recession velocity is calculated by either the classical relation, $v_c = z\times c$, or the relativistic relation, $v_r = \frac{(1+z)^2-1}{(1+z)^2+1}c$. The calculations for the three galaxies are summarized in the following Table:

Galaxy	v_c (km/s)	v_r (km/s)	$v_r/c\times 100$	$r = v_r/H_o$ (Mpc)
3C279	160800	121390	40%	1660
3C245	308700	182740	61%	2500
4C41.17	1140000	275040	92%	3770
		(α)	(β)	(γ)

From columns 2 and 3 it is obvious that the relativistic relation should be used. It should be noted that for 3C245 και 4C41.17 the classical relation gives velocities larger than the speed of light. It is also noted that the difference between the distance calculated by the classical relation and the relativistic relation it is about 30%, even for the "nearby" galaxy, 3C 279.

The answers to the questions are given in columns 3, 4 and 5.

16. The absolute visual magnitude of a white dwarf star is $M_V = 10$ mag and its effective temperature $T_{eff} = 10000$ K. Its bolometric correction is $B.C. = 0.6$ mag. Find its radius R as a function of the solar radius.

Answer : The bolometric correction is given by: $B.C. = M_{\mathrm{bol}} - M_V$, therefore the white dwarf's bolometric magnitude is $M_{\mathrm{bol}} = B.C. + M_V$ or $M_{\mathrm{bol}} = 0.6 + 10 = 10.6$ mag

Using the equation: $M_\odot - M_{\mathrm{bol}} = 2.5 \log\left(\frac{L_{bol}}{L_\odot}\right)$ we get:

$$4.8 - 10.6 = 2.5 \log\left(\frac{L_{bol}}{L_\odot}\right) \rightarrow \log\left(\frac{L_{bol}}{L_\odot}\right) = -2.32$$

But : $\frac{L_{bol}}{L_\odot} = \frac{4\pi R^2 \sigma T^4}{4\pi R_\odot^{\ 2} \sigma T_\odot^{\ 4}}$ or $\frac{L_{bol}}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T}{T_\odot}\right)^4$

Therefore : $2\log\left(\frac{R}{R_\odot}\right) + 4\log\left(\frac{T}{T_\odot}\right) = -2.32$ or $\log\left(\frac{R}{R_\odot}\right) + 2\log\left(\frac{10000}{5800}\right) = -1.16$ or

$\log\left(\frac{R}{R_\odot}\right) = -1.64$ and $\left(\frac{R}{R_\odot}\right) = 0.023$ and finally $\boldsymbol{R = 0.023\, R_\odot}$

(Note: If the student uses the old definition $B.C. = M_V - M_{\mathrm{bol}}$, accept it as correct)

17. Find the velocity with which a rocket needs to be launched from Earth in order to escape from the gravitational field of (a) the Earth and (b) the Sun. Assume that the Earth's orbit around the Sun is circular. Ignore the influence of other planets.

Find the minimum launch velocity of the rocket, in order to escape the gravitational field of the Sun, if the launch is performed from the Earth, tangentially to the Earth's orbit around the Sun?

Answer:

(1). (a) The escape velocity $\upsilon_\oplus^{esc}$ of a body from the gravitational field of the Earth, when it is located at the Earth's surface, is given by

$\upsilon_\oplus^{esc} = \sqrt{\frac{2GM_\oplus}{R_\oplus}}$, where $M_\oplus$ is the mass of the Earth, $R_\oplus$ is the radius of the Earth and G is the gravitational constant.

Using the values given in the *Table of Constants*, we derive $\upsilon_\oplus^{esc} = 11.7\, km/s$ **(2 points)**

(b) The escape velocity $\upsilon_\odot^{esc}$ of a body, located on the Earth's orbit, from the gravitational field of the Sun, is given by

$\upsilon_{\odot}^{esc} = \sqrt{\dfrac{2GM_{\odot}}{a_{\oplus}}}$, where $M_{\odot}$ is the Sun's mass, $a_{\oplus}$ is the average Earth-Sun distance and G is the gravitational constant.

(Proof: $\dfrac{1}{2}m(\upsilon_{\odot}^{esc})^2 = \dfrac{GmM_{\odot}}{a_{\oplus}} \Rightarrow \upsilon_{\odot}^{esc} = \sqrt{\dfrac{2GM_{\odot}}{a_{\oplus}}}$)

Using the values given in the *Table of Constants*, we derive $\upsilon_{\odot}^{esc} = 42.3 km/s$ **(2 points)**

The magnitude of the total escape velocity is given by (note that the two components are perpendicular) $\upsilon_{o\lambda}^{esc} = \sqrt{\left(\upsilon_{\oplus}^{esc}\right)^2 + \left(\upsilon_{\odot}^{esc}\right)^2} \approx 44 km/s$ **(4 points)**

(2). The magnitude of the rotational velocity of the Earth around the Sun, $\upsilon_{\oplus}$, is approximately

$$\upsilon_{\oplus} = \sqrt{\frac{GM_{\odot}}{a_{\oplus}}} = 29.9 km/s$$ **(2 points)**

(Proof: $\alpha_{centrifugal} = \alpha_{gravity} \Rightarrow \dfrac{\upsilon_{\oplus}^2}{a_{\oplus}} = \dfrac{GM_{\odot}}{a_{\oplus}^2} \Rightarrow \upsilon_{\oplus} = \sqrt{\dfrac{GM_{\odot}}{a_{\oplus}}}$)

Therefore if the rocket is launched tangentially to the Earth's orbit, the escape velocity from the gravitational pull of the sun will differ, from 42.3km/s to (42.3 – 29.9) km/s, i.e. 12.4km/s.**(2 points)**

Marking scheme for deduction of points for decimal point, etc

$(I(r) = I(0) \times e^{-\tau})$

Theoretical Exam - Long Questions

Question 1

In a homogeneous and isotropic universe, the matter (baryonic matter + dark matter) density parameter $\Omega_m = \frac{\rho_m}{\rho_c}$ = 32%, where ρ_m is the matter density and ρ_c is the critical density of the Universe.

(1) Calculate the average matter density in our local neighbourhood.

(2) Calculate the escape velocity of a galaxy 100 Mpc away from us. Assume that the recession velocity of galaxies in Hubble's law equals the corresponding escape velocity at that distance, for the critical density of the Universe that we observe.

(3) The particular galaxy is orbiting around the centre of our cluster of galaxies on a circular orbit. What is the angular velocity of this galaxy on the sky?

(4) Will we ever discriminate two such galaxies that are initially at the same line of sight, if they are both moving on circular orbits but at different radii (answer "Yes" or "No")? [Assume that the Earth is located at the centre of our local cluster.]

Answer:

(1) The critical density $\rho_c = \frac{3H_0^2}{8\pi G}$ **(9 points)**

If the matter density parameter $\Omega_m = \frac{\rho_m}{\rho_c}$ is 32% , thus $\rho_m = 0.32\frac{3H_0^2}{8\pi G}$. **(3 points)**

From the latest estimate of $H_0 = 67.8 \frac{km\,s^{-1}}{Mpc}$ we obtain $\rho_m = 8.6\times10^{-27}$ kg m^{-3} **(4 points)**

(2) The escape velocity is $\upsilon_{esc} = \sqrt{\frac{2GM}{d}}$. **(4 points)**

By replacing $M = \rho_c \frac{4\pi}{3} d^3$ **(2 points)**

we obtain, for the escape velocity within d = 100 Mpc

$$\upsilon_{esc} = \sqrt{\frac{8\pi}{3} G \times 0.32 \times \frac{3\mathrm{H}_0^2}{8\pi G}} \times 100\,\mathrm{M}pc = 3835 \text{ km s}^{-1}$$ **(8 points)**

(3) If a galaxy is orbiting around the centre of our galaxy, its velocity is $\frac{1}{\sqrt{2}}$ of its escape velocity. Thus

$$\omega = \frac{v}{d} = \frac{v_{esc}/\sqrt{2}}{d} = \frac{H_0 d\sqrt{\Omega_m/2}}{d} = \frac{\sqrt{0.32}\,(67.8\times10^3 m/s)/(3.09\times10^{22} m)}{\sqrt{2}} = 8.8\times10^{-19} rad/s$$ **(12 Points)**

This is

$1.8\times10^{-13} arc\sec/s$ and it does not depend on the distance d.

(4) Therefore we will never be able to resolve them and the answer is "No". **(8 Points)**

Question 2

A spacecraft is orbiting the Near Earth Asteroid (2608) *Seneca* (staying continuously very close to the asteroid), transmitting pulsed data to the *Earth*. Due to the relative motion of the two bodies (the asteroid and the Earth) around the *Sun*, the time it takes for a pulse to arrive at the ground station varies approximately between 2 and 39 minutes. The orbits of the Earth and *Seneca* are coplanar. Assuming that the *Earth* moves around the *Sun* on a circular orbit (with radius a_{Earth} = 1 AU and period T_{Earth} = 1 yr) and that the orbit of *Seneca* does not intersect the orbit of the *Earth*, calculate:
(1) the semi-major axis, a_{Sen} the eccentricity, e_{Sen} of *Seneca's* orbit around the *Sun*
(2) the period of *Seneca's* orbit, T_{Sen} and the average period between two consecutive oppositions, T_{syn} of the *Earth-Seneca* couple
(3) an approximate value for the mass of the planet Jupiter, M_{Jup} (assuming this is the only planet of our Solar system with non-negligible mass compared to the Sun). Assume that the presence of Jupiter does not influence the orbit of Seneca.

Answer :
(1) For Δt_b = 2 min = 120 sec, the distance travelled by a light pulse is $R_1 = c\times\Delta t$ or R_1 = 0.24 AU, while for Δt_a = 39 min the maximum distance is $R_2 = c\times\Delta t$ or R_2 = 4.67 AU. **(4 points)**

Since the orbits do not intersect and the R_2 exceeds by far 1 AU, the orbit of *Seneca* is exterior to that of the *Earth*. R_1 corresponds to the minimum relative distance of the two bodies (i.e. at opposition), while R_2 corresponds to the maximum relative distance (i.e. at conjunction).

If $q = a\times(1–e)$ is the perihelion and $Q = a\times(1+e)$ the aphelion distance of *Seneca*, then $R_1 = q–1$ AU (the minimum distance of *Seneca* from the *Sun* minus the semi-major axis of the *Earth's* orbit), while $R_2 = Q+1$ AU (the maximum distance of *Seneca* from the *Sun* plus the semi-major axis of the *Earth's* orbit). **(6 points)**
Thus,

$a_{\text{Sen}}\,(1–e_{\text{Sen}}) = 1+R_1 = 1.24$ AU
$a_{\text{Sen}}\,(1+e_{\text{Sen}}) = R_2–1 = 3.67$ AU   from which one finds $\boldsymbol{a_{\text{Sen}}} \sim 2.46$ AU **(2 point)**
and $\boldsymbol{e_{\text{Sen}}} \sim 0.49$ **(2 points)**

(figures rounded to 2 dec. digits)

(2) The period, T_{Sen}, of *Seneca's* orbit can be found by using Kepler's 3rd law for *Seneca* and the *Earth* (ignoring their small masses, compared to the Sun's):

$a_{\text{Sen}}^{\ 3} / T_{\text{Sen}}^{\ 2} = a_{\text{Earth}}^{\ 3} / T_{\text{Earth}}^{\ 2} = 1$ **(6 points)**

from which one finds $\boldsymbol{T_{\text{Sen}}} \sim 3.87$ yr **(2 point)**

(Alternatively one can use Kepler's law for *Seneca* only and use natural units for the mass of the *Sun* and G, i.e. [mass] = 1 M_{Sun}, [t]= 1 yr and [r] = 1 AU, in which case $\boldsymbol{T_{\text{Sen}} = a_{Sen}^{3/2}}$ = 3.86 yr)
(8 points)

Assuming non-retrogate orbit, the synodic period, T_{syn} of *Seneca* and *Earth* is given by

$1/T_{syn} = 1/T_{Earh} - 1/T_{Sen}$ (*Seneca* is superior) **(6 points)**

which gives $\boldsymbol{T_{syn}} = 1.35$ yr **(2 point)**

(3) From Kepler's law, we can calculate the mass of the *Sun*

$4\pi^2 a_{Sen}{}^3 / T_{Sen}{}^2 = G M_{Sun}$, i.e. $M_{Sun} \sim 1.988\times10^{30}$ kg (depending on accuracy) **(4 points)**

Then, using Kepler's 3rd law for *Seneca* and *Jupiter*, one gets

$(a_{Jup}{}^3 / T_{Jup}{}^2) / (a_{Sen}{}^3 / T_{Sen}{}^2) = (M_{Sun} + M_{Jup}) / M_{Sun} = 1 + x$ **(4 points)**

where $x =$ is the mass ratio of *Jupiter* to the *Sun*. Then, solving for x we get (depending on the accuracy used in determining the elements of *Seneca's* orbit)

$x = 0.0016$, thus $M_{Jup} = 3.2\times10^{27}$ kg **(4 points)**

Question 3

(1) Using the *virial theorem* for an isolated, spherical system, i.e, that $-2\langle K\rangle=\langle U\rangle$, where "$K$" is the average kinetic energy and "U" is the average potential energy of the system, determine an expression for the total mass of a cluster of galaxies if we know the radial velocity dispersion, σ, of the cluster's galaxy members and the cluster's radius, R. Assume that the cluster is isolated, spherical, has a homogeneous density and that it consists of galaxies of equal mass.
(2) Find the *virial mass*, i.e. the mass calculated from the *virial theorem*, of the Coma cluster, which lies at a distance of 90 Mpc from us, if you know that the radial velocity dispersion of its member galaxies is $\sigma_{\upsilon_r} = 1000 km/s$ and that its angular diameter (on the sky) is about 4^o.
(3) From observations, the total luminosity of the galaxies comprising the cluster is approximately $L = 5\times10^{12}\ L_\odot$. If the mass to luminosity ratio, M/L, of the cluster is ~1 (assume that all the mass of the cluster is visible mass), this should correspond to a total mass $M \sim 5\times10^{12}\ M_\odot$ for the mass of the cluster. Give the ratio of the luminous mass to the total mass of the cluster you derived in question (2).

Answer:

(1) Using the virial theorem for our isolated, spherical system of N galaxies of mass m, each, we get

$$-2\langle K\rangle = \langle U\rangle \Rightarrow -\frac{2}{N}\sum_1^N \frac{1}{2} m_i u_i^2 = < U > \Rightarrow -\frac{m}{N}\sum_1^N u_i^2 = \frac{U}{N}$$ **(3 point)**

where $\frac{1}{N}\sum_1^N u_i^2 \approx \langle u^2\rangle = \langle u_r{}^2\rangle + \langle u_\theta{}^2\rangle + \langle u_\phi{}^2\rangle$, where u_r, u_θ and u_φ are the radial velocity and the two perpendicular velocities on the plane of the sky of the members of the cluster. **(6 points)**

Assuming that $\langle u_r^2 \rangle \sim \langle u_\theta^2 \rangle \sim \langle u_\phi^2 \rangle$, we have

$$-3m\langle u_r^2 \rangle = U/N = (-\frac{3}{5}\frac{GM^2}{R})/N = -\frac{3}{5}\frac{GMm}{R} \Rightarrow M \approx \frac{5R\langle u_r^2 \rangle}{G}$$ **(13 points)**

(where we used the gravitational potential of a spherical homogeneous mass M enclosed within radius R)

Alternatively the student can give a rougher order of magnitude estimate

$-2\langle K \rangle = \langle U \rangle \Rightarrow -2.\frac{1}{2}\langle u_r^2 \rangle \approx -\frac{GM}{R}$ etc If they do not use the exact formula: (11 Points)

(2) From the result of the previous question we have $M \approx \frac{5R\langle u_r^2 \rangle}{G}$, where

$\sqrt{\langle u_r^2 \rangle} = \sigma = 1000\ km\, s^{-1}$ **(8 points)**

The angular diameter of the cluster is $\varphi = 4^o$ at a distance of d = 90 Mpc. Therefore the diameter D of the cluster in Mpc is calculated from:

$$\tan\phi = \frac{D}{d} \Rightarrow \phi(rad) \approx \frac{D}{d} \Rightarrow D \approx 4 \times \frac{\pi}{180} \times 90\,\mathrm{M}pc \approx 6.3Mpc \Rightarrow R \approx 3Mpc$$ **(11 points)**

Therefore $M \approx \frac{5R\langle u_r^2 \rangle}{G} = 3.6 \times 10^{15}\mathrm{M}_\odot$ **(3 point)**

(3) $\frac{M_{virial}}{L_{galaxies}} = \frac{3.6 \times 10^{15}\mathrm{M}_\odot}{5 \times 10^{12} L_\odot} = 720\frac{\mathrm{M}_\odot}{L_\odot}$ This is obviously much larger from $\frac{\mathrm{M}_\odot}{L_\odot} = 1$ which is found from the visible mass of the cluster. Thus $\frac{M}{M_{virial}} = \frac{1}{720}$. **(6 points)**