Data Analysis: Instructions

- **Do not touch envelopes until the start of the examination.**
- The data analysis examination lasts for 3 hours and is worth a total of 125 marks.
- There are **Answer Sheets** for carrying out detailed work and **Working Sheets** for rough work, which are already marked with your student code and question number.
- ***Use only the answer sheets for a particular question for your answer. Please write only on the printed side of the sheet. Do not use the reverse side.*** If you have written something on any sheet which you do not want to be evaluated, cross it out.
- Use as many mathematical expressions as you think may help the evaluator to better understand your solutions. The evaluator may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
- You are not allowed to leave your work desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, etc.), please draw the attention of the supervisor.
- The beginning and end of the examination will be indicated by the supervisor. The remaining time will be displayed on a clock.
- At the end of the examination you must stop writing immediately. Put everything back in the envelope and leave it on the table.
- Once all envelopes are collected, your student guide will escort you out of the examination room.
- A list of constants and useful relations are included in the envelope.

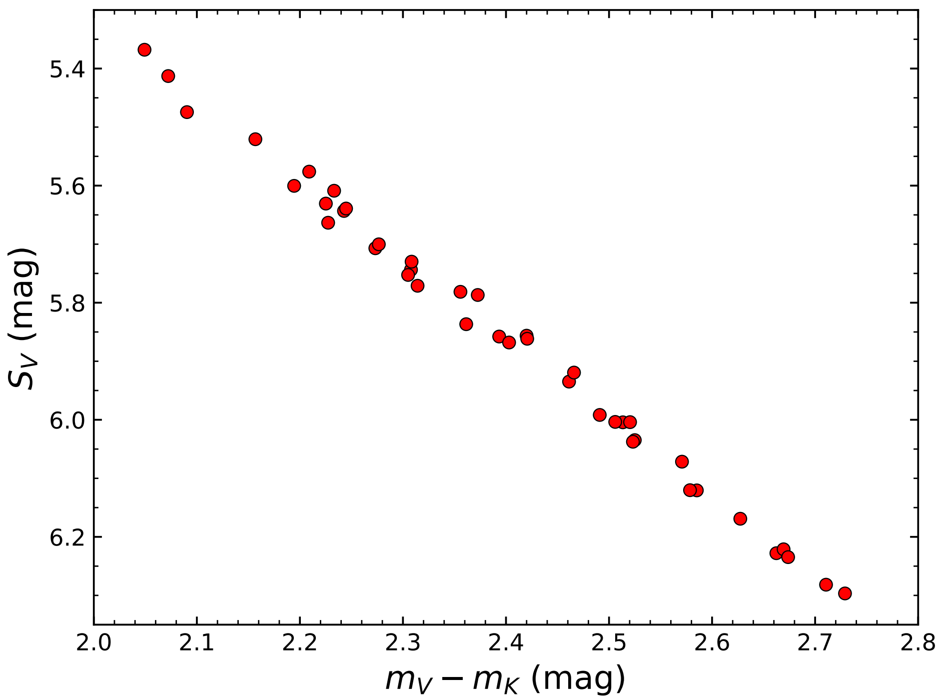
Data Analysis 1: 'Distance to the Large Magellanic Cloud'

In 2019 an international collaboration led by Polish astronomers measured, with very high precision and accuracy, the distance to the Large Magellanic Cloud (LMC), a satellite galaxy of the Milky Way. In this way they set the zero point of the extragalactic distance scale, which allowed for a very precise measurement of the Hubble constant. Their method involved measuring the distances to 20 eclipsing binary stars in the LMC, using the concept of the surface brightness S_V of a star defined as:

$$S_V = m_V + 5\log_{10}\theta,$$

where m_V is the magnitude of a star in the optical V band and θ is the angular diameter of the star on the sky in milliarcseconds (mas).

The quantity S_V can be understood as the magnitude of a star with an angular diameter of 1 mas. An empirical relation has been established between S_V and the colour index $(m_V - m_K)$, where m_V and m_K are magnitudes in the V-band and infrared K-band. This is shown in the figure below for giant stars of spectral types G and K.



Using this relation, the distance to an eclipsing binary system can be determined by deriving the physical radii of the components (using photometry and spectroscopy), and comparing these with the angular diameters predicted by the S_V – $(m_V - m_K)$ relation.

The table below gives the parameters of three detached eclipsing binary stars. R_1 and R_2 are the radii of each component, V_{1+2} and K_{1+2} are the total brightness in magnitudes of the binary in the V- and K-bands, and L_2/L_1 is the luminosity ratio of the components in each band.

source ID	R_1 [$R_\odot$]	R_2 [$R_\odot$]	V_{1+2} [mag]	K_{1+2} [mag]	L_2/L_1 (V)	L_2/L_1 (K)
OGLE LMC-ECL-03160	17.03	37.42	16.73	14.10	2.80	4.23
OGLE LMC-ECL-10567	24.60	36.64	16.15	13.83	1.41	1.99
OGLE LMC-ECL-18365	37.30	15.94	16.27	14.01	0.206	0.188

Apply the method outlined above to the three eclipsing binary systems and calculate the distance to the LMC in kiloparsecs. Estimate the total error of the result. Assume that the fitting of the $S_V - (m_V - m_K)$ relation contributes to a bias of up to 0.8% in all measurements simultaneously.

(Total: 50 points)

Hint: in your calculations keep at least three significant figures and two decimal places. Assume that interstellar extinction is negligible and that the angular size of the LMC is small.

Solution

Distance calculation

The information from the table can be used to derive individual magnitudes of both components according to equations:

$$m_1 = m_{1+2} + 2.5 \log_{10}(1 + L_2/L_1)$$

$$m_2 = m_{1+2} + 2.5 \log_{10}(1 + L_1/L_2).$$

We will use the third system (OGLE LMC-ECL-18365) as an example to demonstrate the calculations in detail. The values for this system are as follows:

Magnitudes: $m_{V,1} = 16.47$ mag, $m_{K,1} = 14.20$ mag, $m_{V,2} = 18.19$ mag, $m_{K,2} = 16.01$ mag.
Colours: $(m_V - m_K)_1 = 2.27$ mag, $(m_V - m_K)_2 = 2.18$ mag.

The second step is to determine the surface brightness S_V for both components using the figure. A least square fit to the data in the figure results in a linear function:

$$S_V = 1.346\,((m_V - m_K) - 2.407) + 5.869 \text{ [mag], or}$$

$$S_V = 1.346(m_V - m_K) + 2.629 \text{ [mag]}.$$

Uncertainties of the coefficients are 1.346 ± 0.017 mag and 2.63 ± 0.04 mag.
For this system, this gives $S_{V,1} = 5.69$ mag and $S_{V,2} = 5.57$ mag.

However, participants will have two other ways of determining S_V.
1) The first way is a graphical way by using a ruler and a pencil in order to draw the 'best-fit' line on the figure. Then S_V follows from an intersection of the $x = (m_V - m_K)$ vertical line and the 'best-fit' line.
2) The second way is to determine the coefficients of the best-fit line $y = ax + b$ by using coordinates of two points on the figure. The points should be far from each other.
For example, the coordinates of the second and the penultimate points are: $(x_1, y_1) = (2.07, 5.41)$ and $(x_2, y_2) = (2.71, 6.28)$. This results in:

$$a = \frac{y_2 - y_1}{x_2 - x_1} = 1.36$$

$$b = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1} = 2.60$$

Using these coefficients we have: $S_{V,1} = y = 2.27 \cdot a + b = 5.69$ mag and $S_{V,2} = y = 2.18 \cdot a + b = 5.56$ mag, so within 0.01 mag of the 'precise' results.

The third step is to calculate angular diameters of components by using the equation presented in the problem. By modifying the equation defining S_V we obtain:

$$\theta = 10^{0.2(S_V - m_V)}$$

Subsequently we get: $\theta_1 = 10^{0.2(5.69-16.47)} = 0.00698$ mas and $\theta_2 = 10^{0.2(5.56-18.19)} = 0.00298$ mas.

The fourth step is to calculate the distance to each target. As components form a physical binary, their distances should be very similar. This is an independent check of the method and calculations. As the angles under which we see stellar discs are very small ($\sin\theta \approx \theta$) we can safely use a linear relation between the angular and physical diameters of an object. We therefore calculate the distance D as $D = kR/\theta$, where θ is expressed in mas, R in solar radii and k is a conversion factor. The conversion factor results from a fraction $(2R_\odot/1\text{ kpc})/(1\text{ mas}/1\text{ rad}) = (2R_\odot/1\text{ AU}) = 9.30 \cdot 10^{-3}$.

The distances for the third system are $D_1 = 9.30 \cdot 10^{-3} \cdot 37.30/0.00698 = 49.70$ kpc and $D_2 = 9.30 \cdot 10^{-3} \cdot 15.94/0.00298 = 49.75$ kpc.

We then repeat the calculation following the same scheme for the first and second systems, obtaining for the first system $D_1 = 49.33$ kpc and $D_2 = 49.07$ kpc, and for the second system $D_1 = 49.56$ kpc and $D_2 = 49.30$ kpc. The unweighted mean of all distances is 49.45 kpc.

Uncertainties

'Statistical' part.

The standard deviation of the sample is $s = 0.24$ kpc. The standard error of the mean is $s/\sqrt{6} = 0.09$ kpc.

OR:

The mean distances to the three eclipsing binaries are: 49.73 kpc, 49.20 kpc and 49.443 kpc. The standard deviation is s=0.22 kpc, and the standard error of the mean is $s/\sqrt{3}$=0.13 kpc.

'Systematic' part.

All distances are inversely proportional to angular diameters derived from the $S_V - (m_V - m_K)$ relation. Thus their accuracy is limited by the precision of the relation. That gives the 'irreducible' part of the error: $0.008 \cdot 49.45 = 0.40$ kpc.

Finally: the distance is $49.45 \pm 0.09 \pm 0.40$ kpc; the uncertainty is completely dominated by the precision of the $S_V - (m_V - m_K)$ relation.

Author's suggestion of scoring

A full proper solution is scored with 50 points.

Individual scores:

Correct formulas for individual magnitudes: 4 p
Derivation of V and K magnitudes for 6 components: 6 p
Determination of the trend line of Sv from the figure (by line fitting or by ruler): 7 p
Calculation of the SV quantity for 6 components: 6 p
Correct formula for angular diameter: 2 p
Determination of angular diameters of 6 components: 6 p
Correct formula for the distance calculation 4 p
Calculation of distances to 6 components: 6 p
Calculation of the final distance: 3 p
Errors: 'statistical': 3 p
Errors: 'systematic': 3 p

Data Analysis 2: 'Isolated black hole'

In 2022, two independent groups reported the discovery of an isolated black hole based on observations of the gravitational microlensing event OGLE-2011-BLG-0462. In this problem, we will analyze data from the Hubble Space Telescope to reproduce their findings.

Gravitational microlensing occurs when the light of a distant star (the 'source') is bent and magnified by the gravitational field of an intervening object (the 'lens'). The characteristic angular scale of gravitational microlensing events, called the angular Einstein radius θ_E, depends on the mass M and distance D_ℓ from the Earth to the lens:

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_s - D_\ell}{D_s D_\ell}},$$

where D_s is the distance to the source star. For typical microlensing events observed in the Milky Way, the source stars are in the Galactic bulge, near the Galactic center, so $D_s \approx 8\,\text{kpc}$.

(a) Calculate the angular Einstein radius in milliarcseconds (mas) for an example lens of $1\,M_\odot$ located at a distance of 1 kpc. (2 points)

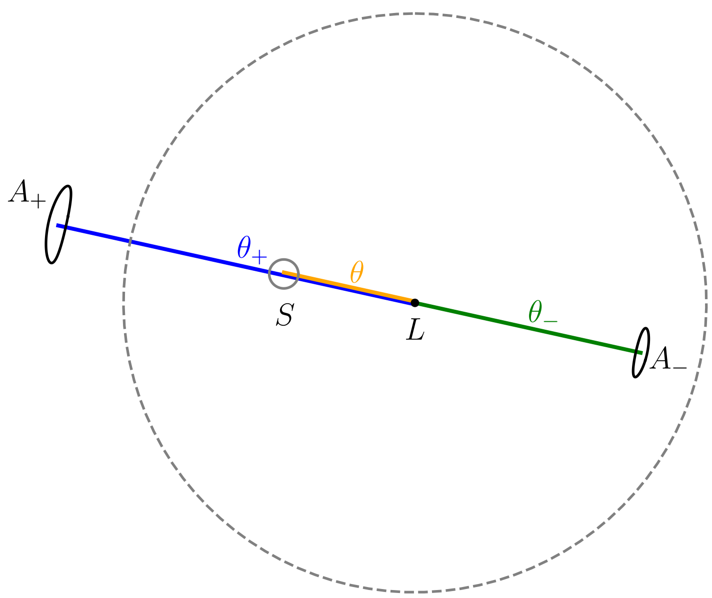
Suppose that at time t the lens and the source are separated by an angle $\theta \equiv u(t)\theta_E$ on the sky. Two images of the source are created on a line through the positions of the source and the lens, at angular distances θ_+ and θ_- from the lens given by:

$$\theta_\pm = \frac{1}{2}\left(u \pm \sqrt{u^2+4}\right)\theta_E.$$

These two images are magnified, relative to the unlensed brightness of the source. The absolute magnification of the images is:

$$A_\pm = \frac{1}{2}\left(\frac{u^2+2}{u\sqrt{u^2+4}} \pm 1\right).$$

The image below shows the geometry of the event. The position of the lens is marked as L, the unlensed position of the source is marked as S, while A_+ and A_- mark the positions of the two images of the source. The dashed circle has a radius of one Einstein radius.

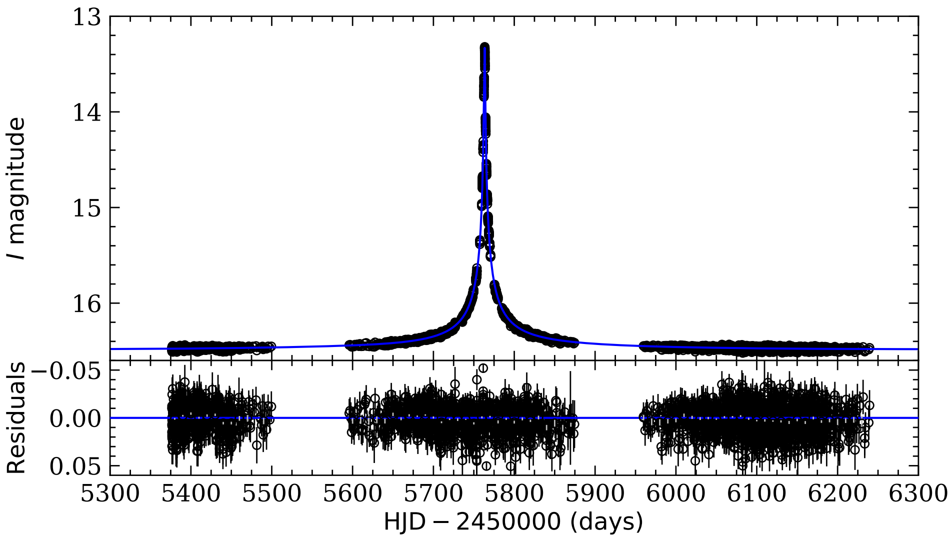


(b) Current telescopes cannot normally resolve this pair of images, but only measure the position of the image centroid, i.e. the brightness-weighted mean of the positions of the two images. Derive an expression for the angular separation θ_c of the image centroid relative to the lens as a function of u and $\theta_{\rm E}$. (8 points)

(c) Derive an expression for the source deflection $\Delta\theta$, i.e. the difference between the location of the centroid and the unlensed position of the source, as a function of u and $\theta_{\rm E}$. What is the source deflection when the lens and the source are nearly perfectly aligned ($u \approx 0$)? (4 points)

The source and lens are moving relative to each other in the sky. Thus, both the total magnification of the images and the position of the centroid changes with time, resulting in observable photometric and astrometric microlensing effects. For now, we assume that the source-lens relative motion is rectilinear.

The plot below shows the light curve of the gravitational microlensing event OGLE-2011-BLG-0462, discovered by the OGLE sky survey led by astronomers from the University of Warsaw. The solid line shows the best-fitting light curve model. The Einstein timescale of the event, i.e. the time needed for the source to move by one angular Einstein radius relative to the lens, was $t_{\rm E} = 247$ days. The event peaked on 21 July 2011 (HJD = 2455763). The minimal separation between the lens and the source was $u_0 \approx 0$.



The table below shows the measured positions of the source star against the background objects in the East and North directions based on images from the Hubble Space Telescope.

HJD	E position (mas)	N position (mas)
2455765.2	2.58 ± 0.13	7.29 ± 0.16
2455865.7	2.32 ± 0.12	5.44 ± 0.24
2456179.7	0.46 ± 0.14	1.62 ± 0.08
2456195.8	0.88 ± 0.36	1.56 ± 0.77
2456426.2	-1.02 ± 0.21	-0.94 ± 0.12
2456587.7	-2.04 ± 0.07	-1.88 ± 0.40
2456956.6	-4.54 ± 0.25	-5.16 ± 0.29
2457995.2	-11.14 ± 0.12	-15.14 ± 0.17

(d) Plot the measured positions of the source star against the background objects in the East and North directions as a function of time. (10 points)

(e) The observed motion of the source star is the sum of two effects: rectilinear proper motion of the source and astrometric microlensing effects. Calculate the proper motion (in mas/year) of the source and its uncertainty in the East and North directions. (8 points)

(f) After subtracting the effects of proper motion from the data, calculate and plot the total resultant astrometric deflection as a function of u. Neglect the uncertainty of the proper motion determination. (20 points)

(g) Analyse the data to determine the angular Einstein radius θ_E of the event and its uncertainty. (Hint: it may be helpful to linearise the expression for $\Delta\theta$).

(16 points)

(h) For long-timescale events such as OGLE-2011-BLG-0462, the rectilinear approximation of the relative lens-source proper motion is not strictly true and the orbital motion of the Earth has to be taken into account. This allows measurement of a dimensionless quantity called the microlensing parallax, defined as $\pi_\mathrm{E} = (\pi_l - \pi_s)/\theta_\mathrm{E}$, where π_l and π_s are parallaxes of the lens and the source, respectively.

For this event $\pi_\mathrm{E} = 0.095 \pm 0.009$. Rearrange the expression for θ_E given earlier to calculate the mass of the lens in solar masses and its uncertainty.

(7 points)

(Total: 75 points)

Solution

(a)

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_s - D_\ell}{D_s D_\ell}} = \sqrt{\frac{4GM}{\mathrm{au}c^2}\left(\frac{\mathrm{au}}{D_\ell} - \frac{\mathrm{au}}{D_s}\right)} = 2.7\,\mathrm{mas}$$

The angular resolution of modern large ($D \approx 10\,\mathrm{m}$) optical ($\lambda = 550\,\mathrm{nm}$) telescopes is $\theta_0 = 1.22\lambda/D \approx 14\,\mathrm{mas}$. Thus, $\theta_0 \gg \theta_{\rm E}$, so the images created during microlensing events cannot be resolved by these telescopes.

(2 points)

(b)

$$\theta_c = \frac{\theta_+ A_+ + \theta_- A_-}{A_+ + A_-} = \frac{\frac{1}{4}\left(u+\sqrt{u^2+4}\right)\left(\frac{u^2+2}{u\sqrt{u^2+4}}+1\right) + \frac{1}{4}\left(u-\sqrt{u^2+4}\right)\left(\frac{u^2+2}{u\sqrt{u^2+4}}-1\right)}{\frac{u^2+2}{u\sqrt{u^2+4}}}\theta_{\rm E}$$

$$= \frac{\left(u+\sqrt{u^2+4}\right)\left(u^2+2+u\sqrt{u^2+4}\right) + \left(u-\sqrt{u^2+4}\right)\left(u^2+2-u\sqrt{u^2+4}\right)}{4(u^2+2)}\theta_{\rm E}$$

$$= \frac{2u(u^2+2)+2u(u^2+4)}{4(u^2+2)}\theta_{\rm E} = \frac{2u(2u^2+6)}{4(u^2+2)}\theta_{\rm E} = \frac{u(u^2+3)}{u^2+2}\theta_{\rm E}$$

(8 points)

(c)

$$\Delta\theta = \theta_c - \theta = \frac{u(u^2+3)}{u^2+2}\theta_{\rm E} - u\theta_{\rm E} = \frac{u(u^2+3) - u(u^2+2)}{u^2+2}\theta_{\rm E} = \frac{u}{u^2+2}\theta_{\rm E}$$

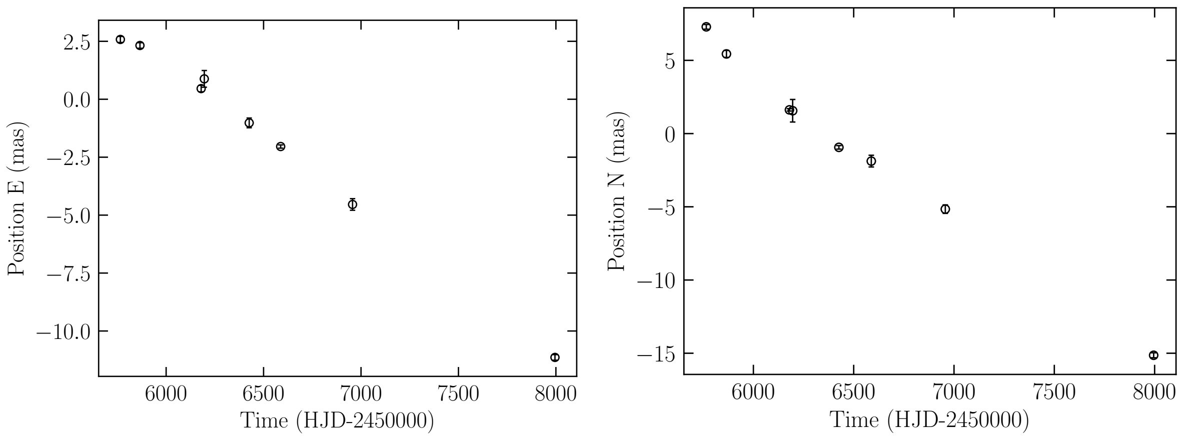
(3 points)

$$\Delta\theta(u=0) = 0$$

so there is no deflection when the lens and the source are nearly perfectly aligned.

(1 points)

(d)



(10 points, 5 points for each graph)

(e) I will use the fact that the first epoch of astrometric observations was taken close to the peak of the light curve (that is, $u_1 \approx 0$, that is, almost no astrometric deflection). Similarly, astrometric deflection is close to zero for the last epoch. Thus,

$$\mu_E \approx \frac{x_{E,8} - x_{E,1}}{t_8 - t_1} = -2.247 \pm 0.029\,\mathrm{mas/yr}$$
$$\mu_N \approx \frac{x_{N,8} - x_{N,1}}{t_8 - t_1} = -3.674 \pm 0.038\,\mathrm{mas/yr}$$

(8 points)

No points should be given if a student does not recognize that the deflection is zero during the first epoch, e.g., they try fitting a straight line to all data points.

(f) I fitted a straight line joining the first and last epoch data and then subtracted it from astrometric measurements. This is because the observed path of the source on the sky is the sum of two effects: the rectilinear proper motion of the source and the astrometric deflection:

$$x(E) = x_1^E + (t - t_1)\mu_{\mathrm{E}} + \Delta\theta(E)$$
$$x(N) = x_1^N + (t - t_1)\mu_{\mathrm{N}} + \Delta\theta(N),$$

where x_1^E is the East position of the source during the first epoch, x_1^N is the North position of the source during the first epoch, t_1 is the time of the first observation, and $\Delta\theta(E)$ and $\Delta\theta(N)$ is the astrometric deflection due to microlensing in East and North direction, respectively.

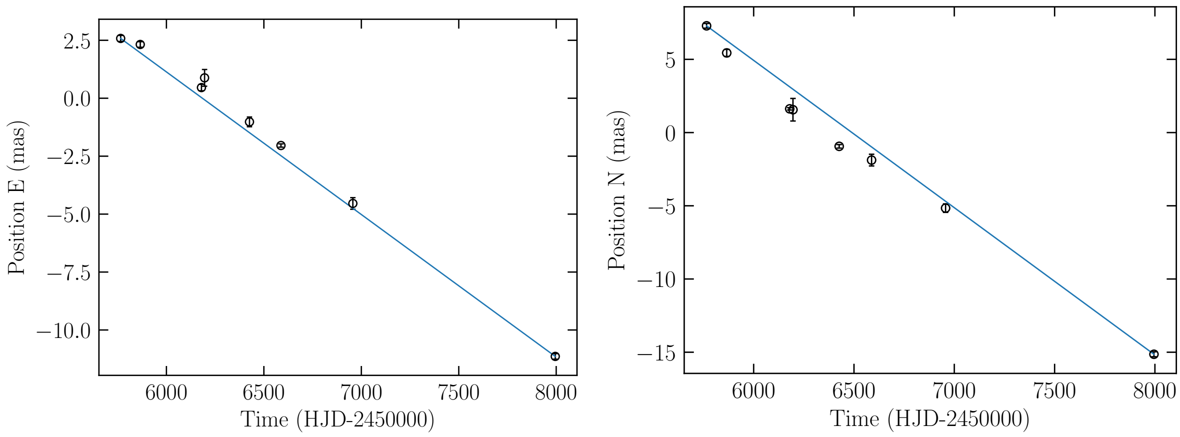


Figure 7: The blue lines join the first and the last epoch data points. The source would move along these lines if there wasnt't any black hole in front of it. However, the position of the source that was observed is deflected due to microlensing effects by the black hole.

Thus, the astrometric deflection during ith epoch is:

$$\Delta\theta(E)_i = x_i^E - x_1^E - (t_i - t_1)\mu_{\mathrm{E}}$$
$$\Delta\theta(N)_i = x_i^N - x_1^N - (t_i - t_1)\mu_{\mathrm{N}}$$

and the total deflection is:

$$\Delta\theta_i = \sqrt{\Delta\theta(E)_i^2 + \Delta\theta(N)_i^2}.$$

I will also use the fact that $u_0 \approx 0$, so $u = (t - t_0)/t_E$, where $t_0 = 2455763$ is the peak time. Results of my calculations are shown in the table below.

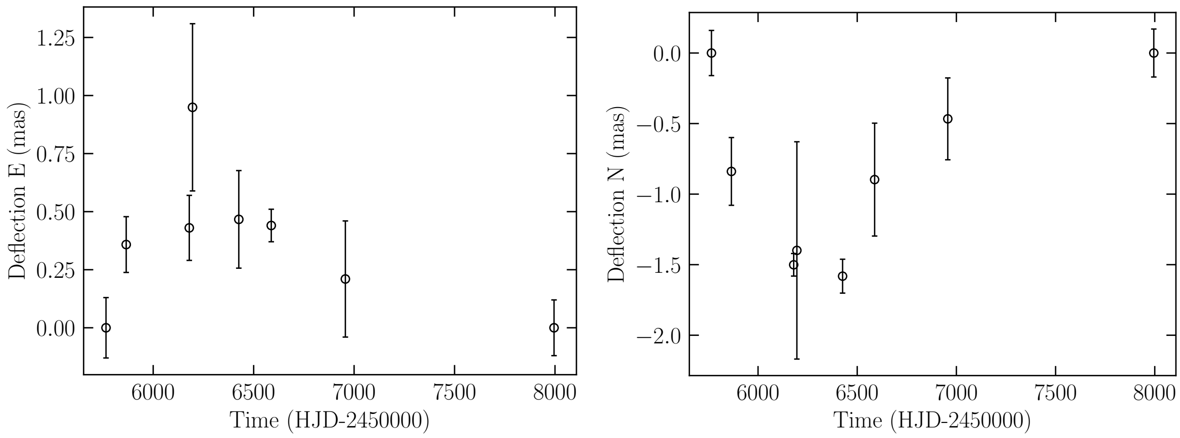


Figure 8: These figures show the astrometric deflection in East and North directions induced by the black hole. (Students are not required to make these plots.)

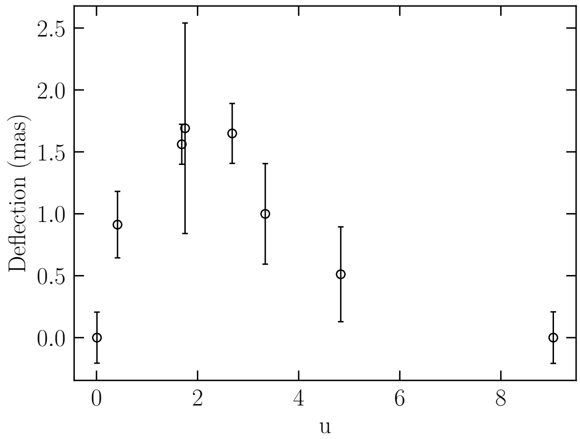


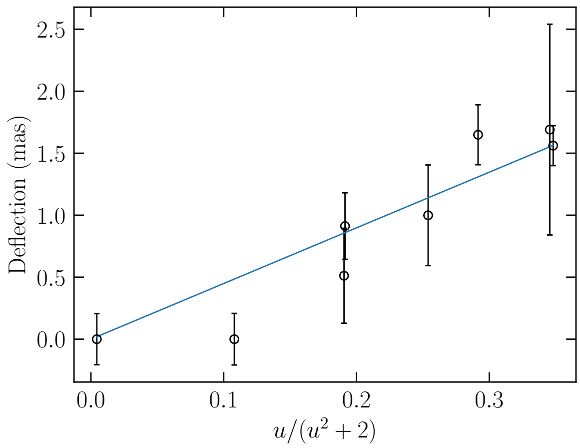
Figure 9: Total astrometric deflection as a function of u. Students are asked to make this plot in part (f).

Epoch	u	$\Delta\theta$ (E,mas)	$\Delta\theta$ (N,mas)	$\Delta\theta$ (mas)
1	0.01	0.00 ± 0.13	-0.00 ± 0.16	0.00 ± 0.21
2	0.42	0.36 ± 0.12	-0.84 ± 0.24	0.91 ± 0.27
3	1.69	0.43 ± 0.14	-1.50 ± 0.08	1.56 ± 0.16
4	1.75	0.95 ± 0.36	-1.40 ± 0.77	1.69 ± 0.85
5	2.69	0.47 ± 0.21	-1.58 ± 0.12	1.65 ± 0.24
6	3.34	0.44 ± 0.07	-0.90 ± 0.40	1.00 ± 0.41
7	4.83	0.21 ± 0.25	-0.47 ± 0.29	0.51 ± 0.38
8	9.04	0.00 ± 0.12	-0.00 ± 0.17	0.00 ± 0.21

(20 points)

(g) We would like to fit the function $\Delta\theta = \frac{u}{u^2+2}\theta_{\rm E}$ to the data. Thus, my "new" independent variable would be $x' = u/(u^2+2)$. Now, I would like to fit the function $y' = \theta_{\rm E} x'$, where $y' = \Delta\theta$. Thus

$$\theta_{\rm E} = \frac{\sum y'_i x'_i/\sigma_i^2}{\sum x'^2_i/\sigma_i^2} \pm \frac{1}{\sqrt{\sum x'^2_i/\sigma_i^2}} = 4.5 \pm 0.4\,\mathrm{mas}$$



Alternative solution:

The gravitational deflection is described by the formula $\Delta\theta = \frac{u}{u^2+2}\theta_{\rm E}$. It can be demonstrated that this function reaches a local maximum for $u = \sqrt{2}$ with $\Delta\theta_{\max} = \frac{\sqrt{2}}{4}\theta_{\rm E} = 0.354\theta_{\rm E}$. From the data, we can estimate the maximum deflection of $\Delta\theta_{\max} = 1.59 \pm 0.14\,$mas. Thus, $\theta_{\rm E} = 4.5 \pm 0.4\,$mas.

(16 points)

(h)

Let $\pi_{\rm rel} = \pi_l - \pi_s$ be the relative lens–source parallax. From the definition of the angular Einstein radius we have

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_s - D_l}{D_s D_l}} = \sqrt{\frac{4GM}{c^2}\left(\frac{1}{D_l} - \frac{1}{D_s}\right)} = \sqrt{\frac{4GM}{c^2 \mathrm{au}}\left(\frac{\mathrm{au}}{D_l} - \frac{\mathrm{au}}{D_s}\right)} = \sqrt{\frac{4GM\pi_{\rm rel}}{c^2\mathrm{au}}}.$$

From the definition of the microlensing parallax $\pi_{\rm rel} = \pi_{\rm E}\theta_E$, so this equation becomes:

$$\theta_{\rm E} = \sqrt{\frac{4GM\pi_{\rm E}\theta_{\rm E}}{c^2\mathrm{au}}},$$

and hence

$$M = \frac{\theta_{\rm E}c^2 AU}{4G\pi_{\rm E}} = 5.8 \pm 0.8 M_\odot.$$

The unceraínty on M can be determined from the relation:

$$\frac{\Delta M}{M} = \sqrt{\left(\frac{\Delta\theta_E}{\theta_{\rm E}}\right)^2 + \left(\frac{\Delta\pi_{\rm E}}{\pi_{\rm E}}\right)^2}$$

(7 points)

Grading

a	Correct value of θ_{E}	2
b	Correct formula	8
c	Correct formula	4
d	Correct plots (5 pts for each plot)	10
e	Correct μ_E within 3σ	3
	Correct μ_E within 5σ	1
	Incorrect μ_E ($> 5\sigma$)	0
	Uncertainty on μ_E	1
	Correct μ_N within 3σ	3
	Correct μ_N within 5σ	1
	Incorrect μ_N ($> 5\sigma$)	0
	Uncertainty on μ_N	1
	No points should be given if the student does not recognize that the deflection is zero during the first epoch	
f	Calculation of the impact parameter u for all epochs	4
	Calculation of the total deflection	6
	Calculation of the uncertainties	5
	Graph	5
	Grant full points if results are correct for 7 or 8 epochs	
	Grant 60% points if results are correct for 5 or 6 epochs	
	Grant 0 points if results are correct for 4 or less than 4 epochs	
g	Correct result with an estimate of the uncertainty	16
	Correct results without the estimate of the uncerainty	8
h	Correct result	7
	TOTAL	75

Grading of the graphs:

- Students can get 5 pts for each correct graphs
- data points with error bars 3 pts
- axis labels with units, tick labels - 1 pts
- graph is clear, fills the entire area - 1 pts

Grant full points if the graph shows correct data for 7 or 8 epochs. Grant 60% points if the graph shows correct data for 5 or 6 epochs. Grant 0 points if the graph shows 4 or less than 4 epochs.