16 August 2022

IOAA

Georgia

Theoretical Competition

Cover Sheet

1 Planck's units (10 points)

In Max Planck's system of natural units of measurements, commonly used in cosmology, all units are expressed in terms of 4 fundamental constants: speed of light (c), universal gravitational constant (G), reduced Planck constant ($\hbar = \frac{h}{2\pi}$) and Boltzmann constant (k_B). Also, ($4\pi\epsilon_0$) is included in this list of constants.
Using dimensional analysis, find expressions for

- Planck length (L_p),
- Planck time (t_p),
- Planck mass (m_p),
- Planck temperature (T_p) and
- the Planck charge (q_p).

(10pt)

2 Circumbinary planet (10 points)

A stellar system consists of two main-sequence stars each of $2M_\odot$ orbiting with a period of 4 years in a circular orbit. A circumbinary planet orbits the center of the binary system in the same orbital plane and direction at a fixed distance of 20 au.

(a) Calculate the amount of time it takes for the planet to reappear at the same position relative to the binary system. **(6pt)**

(b) At a given instant, what is the maximum fraction of the planet's surface area which receives light from at least one of the stars? Neglect any atmospheric effects and the sizes of the stars. **(4pt)**

3 Expanding ring nebula (10 Points)

A planetary nebula, located 100 pc from the Earth, has the shape of a perfect circular ring with an inner radius of $7.0'$ and an outer radius of $8.0'$. Its luminosity is powered by the UV radiation from the white dwarf remnant at the center of the nebula. From other observations, we know that 2000 years ago the inner radius of the nebula was $3.5'$ and outer radius was $4.0'$. We believe that throughout these 2000 years the evolution of the nebula obeys a free expansion scenario, so gravity is negligible and the expansion velocity remains constant with time. Assume that all material in the planetary nebula was ejected at the same instant, but different gas particles have different velocities.

(a) Estimate the range of velocities of the gas particles **(4pt)**

(b) Is the assumption of free expansion justified? Write **YES** or **NO** alongside appropriate calculations. **(3pt)**

(c) If this planetary nebula is bright enough, would an astronaut aboard the International Space Station be able to resolve the thickness of the shell clearly? Write **YES** or **NO** alongside with necessary calculations. **(3pt)**

4 Journey Between Galaxies (10 Points)

We start a long journey to a planet located at a distance of $d_0 = 10\,\mathrm{Mpc}$ at the start of our journey. During our travel, the universe keeps expanding according to Hubble-Lemaître's law. (Assume that Hubble's constant (H) does not change.)

(a) Write an expression for the distance between Earth and the planet at a time t from the start of our journey. **(2pt)**

(b) What minimum constant speed v_0 should the rocket have to reach the destination? Assuming that the speed of our rocket is $v_0 = 1000\,\mathrm{km/s}$, can we reach the planet? Write **YES** or **NO**. If yes, how long will our journey take? **(8pt)**
Hint: A simplified way to account for the expansion of our universe is to introduce a "scale factor" $a(t)$, which relates distance between two objects $l(t)$ at a time t to a distance l_0 at a time $t = 0$ as $l(t) = a(t)l_0$

5 Flaring protoplanetary disk (10 Points)

Planets are the products of collapsing clouds forming circumstellar disks around the young stars. In this problem, we examine the thermal structure of a type of protoplanetary disks. We consider radiation from the central star to be the dominant heating process and assume that the disk is optically thick and the radiation can be only absorbed in the surface layers of the disk. This type of disk is called a *flaring* disk (see figure 1 below). In this case, the star illuminates the surface of the disk directly, and the shallow angle β between the light rays and the surface of the disk increases with distance from the star.

(a) Find an expression for β as a function of r. Assume $h(r) \ll r$. **(4pt)**
Hint: the expression involves $h(r)$, r and $\frac{\Delta h(r)}{\Delta r}$

(b) Find an expression for the equilibrium temperature of the disk (T_D) as a function of $\beta(r)$, r and luminosity (L_s) of the star (Ignore the size of the star). **(3pt)**

(c) We parameterize the disc height as $h(r) = ar^b$, where a and b are constants. For which values of a and b is the condition for an isothermal layer (T_D =Constant) satisfied? Write your answer for constant a in terms of T_D and L_S **(3pt)**
Hint: If $\Delta r \ll r$, then $(1 + \frac{\Delta r}{r})^b \approx 1 + \frac{b\Delta r}{r}$.

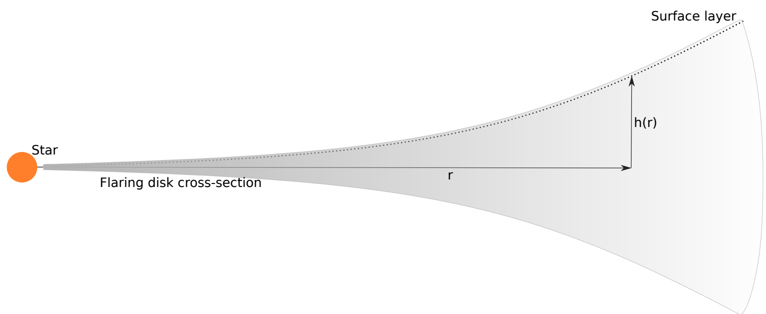


Figure 1: cross section of flaring disk

6 Photometry of Binary stars (20 Points)

We observe a binary system, at a distance $d = 89\,\text{pc}$, with a circular, edge-on orbit around the common center of mass and period of 100 days. Our space-based telescope is of diameter $D = 10\,\text{m}$ and operates at a wavelength $\lambda = 364\,\text{nm}$. We notice that for a total of 38 days during each full period, the two stars cannot be resolved by this telescope as separate objects. The wavelength of peak emission for star A is $\lambda_A = 500\,\text{nm}$ and that for star B is $\lambda_B = 600\,\text{nm}$.

(a) What are the temperatures of the stars A and B, (T_A, T_B) **(2pt)**

(b) Calculate the distance l between the stars. **(6pt)**

(c) Calculate the sum of the masses of the stars (M_T). **(2pt)**

Combined photometry of the system:

	Configuration	U_0	$(U-B)$	$(B-V)$	BC
1	Stars next to each other	6.39	0.2	0.1	0.1
2	B transiting in front of A	6.86	0.25	0.12	0.175

The interstellar extinction per kpc of U is $a_U = 1.4\,\text{mag/kpc}$. The ratio of the densities of the stars $\rho_A/\rho_B = 0.7$
note: For Bolometric correction we use convention:

$$BC = m_{bol} - m_V$$

(d) Calculate the masses of both the stars (M_A, M_B). **(10pt)**

7 Georgia to Georgia (20 points)

Astronomer Keto was flying west overnight along the shortest possible route from Tbilisi (the capital of Georgia) to Atlanta (the capital of the US state of Georgia). She noticed that she is able to observe the star Furud (ζ CMa) throughout the entire flight from one of the jet's windows (although it did touch the horizon at one point, when it also happened to be exactly due South).

Calculate the latitude ϕ_B and longitude λ_B, of Atlanta where she landed. **(20pt)**

It is given that:

- The journey had a duration of 11 hours 25 minutes, and the jet travelled at an average speed of 875 km/h.
- Furud has a declination of $\delta_F = -30°4'$.
- The coordinates of Tbilisi are $\phi_A = 41°43'\text{N}$ and $\lambda_A = 41°48'\text{E}$.
- You should ignore the effect of the rotation of the Earth during the flight, the altitude of the jet, atmospheric refraction, and any wind.

8 Ring of a planet (20 Points)

We are given a flat disk of mass M with inner radius r and the outer radius R.

(a) A point mass m is located on the symmetry axis of the disk at a distance x from the plane of the disk (We assume throughout that mass m stays on the symmetry axis, and is free to move in the vertical direction.). What is the gravitational force exerted on the point mass? **(10pt)**

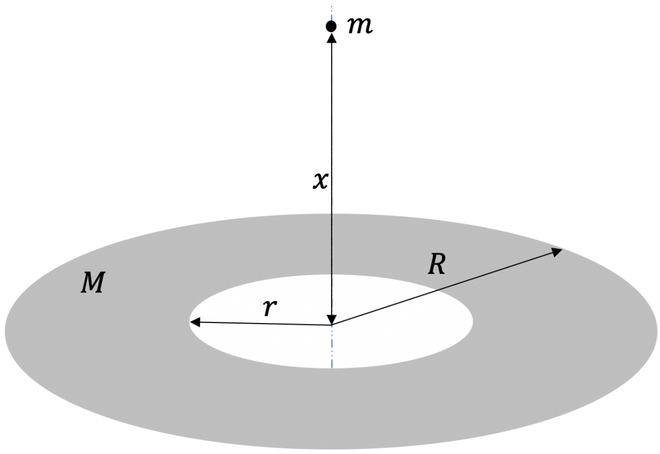


Figure 2: Flat disk around a point mass

Hint 1: You may denote surface density by the symbol σ and calculate force exerted by a small surface element ΔS subtending a solid angle $\Delta\Omega$ with m

Hint 2: The area cut by cone with opening angle 2θ on a sphere of radius R:

$$S = 2\pi R^2(1 - cos(\theta))$$

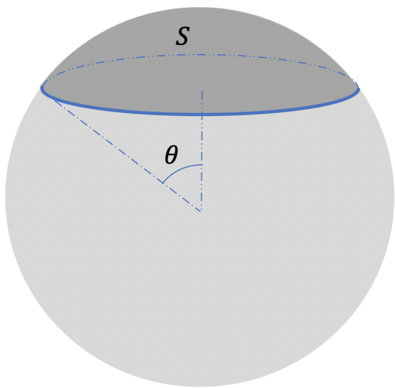


Figure 3: Area cut by a cone on a sphere

(b) what will be the frequency of the small oscillations of the system ($x \ll r$). **(10pt)**

9 Solar Retrograde Motion on Mercury (20 points)

Mercury's orbit has an unusually high eccentricity. Further, its sidereal rotational period is $\frac{2}{3}$ of its sidereal year. As a consequence of these factors, the Sun will exhibit retrograde motion in Mercury's sky, when Mercury is near perihelion.

Calculate the total duration of this apparent solar retrograde motion during one orbit of Mercury around the Sun. Express your answer in Earth days. **(20pt)**

10 Accretion (20 Points)

Consider a compact object (such as black hole, white dwarf or neutron star) with a spherically symmetric accretion of gas, assume that accreting gas is hydrogen. As particles fall into the object they heat up and radiate, thus creating radiation pressure acting on rest of the accreting material. This force is given by,

$$F_L = \sigma_e \frac{I}{c}$$

$$\text{where, } \sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2$$

is the Thompson cross section for electrons, c is the speed of light and I intensity of the light. Although F_L is calculated for electrons, it effectively acts on the whole atom.

(a) If the central compact object has a mass M, find an expression for the Eddington limit (L_E), which is the maximum possible luminosity for the accretion sphere. **(3pt)**

(b) For a particular compact object, the luminosity due to material accretion is the same as the Solar luminosity $L_\odot$. What is its minimum possible mass of this object to achieve this luminosity (in units of $M_\odot$)? **(4pt)**

Assume that the atoms in the accretion sphere originate far away from the compact object. When these atoms fall into the compact object, their gravitational energy is transformed into radiation.

(c) Derive an expression for accretion luminosity (L_{acc}) in terms of the compact object's mass (M), mass accretion rate ($\dot{M} = \frac{\Delta M}{\Delta t}$), and the compact object's radius (R). **(2pt)**

(d) Show that the maximum possible accretion rate in steady state is not directly dependent on the mass of the compact object. **(2pt)**

(e) For an object with $R = 12 \times 10^9$ m, calculate the maximum possible accretion rate $\dot{M}$ (in units of $M_\odot$ per year). **(2pt)**

In reality, the accretion geometry is disk shaped, where most particles trace an almost circular orbit around the compact object. Consider a binary system consisting of a compact object of mass M_1 and a hydrogen burning star of mass M_2 at a distance a from each other. The gas from the hydrogen burning star is accreted by the compact object and due to this mass transfer the period of the binary system changes.

(f) Find the condition on the two masses such that the separation between the stars is increasing. Ignore the rotation of the stars. **(7 pt)**

11 Dyson Sphere (50 Points)

The **Kardashev scale** distinguishes three stages of evolution of civilizations according to the criterion of access to and use of energy.
A type II civilization is capable of harnessing all the energy radiated by its own star. Currently, we are a type zero civilization (we are not even harnessing 100% of the energy that reaches the Earth). One of the ways to become a type II civilization is by building a **Dyson Sphere**. You can imagine it as a sphere, built around the Sun, having the inner surface covered with solar panels.

We assume that modern solar panels are used to build the sphere. First, let's find out at what distance from the Sun should it be built.

Emissivity of the back side of solar panels is $\epsilon = 0.8$.

(a) Solar panels absorb and transfer about $k = 30\%$ of the incident radiation into internal heat. Find the temperature of a Dyson Sphere of radius R. Express your answer in terms of k, R, ϵ and $L_\odot$ (solar luminosity). You may consider the Sun to be a black body. Ignore reflections from the solar panels. Ignore any possible effect of the energy not transferred to the internal heat of the solar panels or into electrical energy generated by the panels. **(3pt)**

Assume that the highest operational temperature for modern solar panels is about $T_{max} \approx 104.5\,°\mathrm{C}$. After that, efficiency drops significantly. To minimize the amount of material used, we should consider building the sphere as small as possible.

(b) Calculate the radius of the sphere for the panels to work properly. Does the Earth stay inside or outside the sphere? Write **IN** or **OUT** in the answer sheet. **(4pt)**

(c) Find the power harnessed by this Dyson Sphere, if the final power output of modern solar panels is about $\eta = 20\%$ of the incident energy. **(2pt)**

(d) Currently, the average power usage of the whole world is about 17 Terawatts. If this Dyson sphere collects energy for one second, for how long could that meet our energy needs? **(2pt)**

(e) In the case where the Dyson sphere completely blocks out the rays of the Sun, the temperature on Earth will drop significantly. Calculate the change in the average temperature of the Earth in this condition, if current average temperature is about $15\,°\mathrm{C}$. Assume that Earth is also a black body. **(5pt)**

(f) Building a rigid spherical object of that size is nearly impossible. Another way of "building" the sphere is by sending individual panels to orbit around the sun (in different inclined orbits) at the radius R found in part b. Calculate the period T of any object orbiting the sun at that radius. **(3pt)**

(g) Assume that each solar panel is a thin sheet of silicon, having unit surface mass $\rho = 1\,\mathrm{kg/m^2}$. The radiation pressure from the Sun, might interfere with the orbit of the panel. Calculate the ratio α of the gravitational and photon forces for unit surface area of panels at distance R. You assume that all incident light is absorbed. Will this radiation pressure have any measurable effect? Write **YES** or **NO** in the Answer Sheet. **(13pt)**

Now assume that the Dyson Sphere is a rigid body rotating with the period found in part 'f' and having the radius found in part 'b'.

(h) A major threat to the Dyson Sphere would be an asteroid whose orbit crosses the surface of the sphere. One way to solve this problem is by removing panels from the path of the asteroid. Obviously the size of the hole through which asteroid should pass is much smaller than radius of the sphere.

Astronomers discovered an asteroid on an orbit in the ecliptic plane in the same direction as the direction of rotation of the Dyson sphere. They calculated that this asteroid will enter the sphere on 14th of August and leave it on 20th of September. Calculate the angular distance between the two holes in the sphere needed to provide a safe passage for the asteroid. **(13pt)**
The trajectory of this asteroid in heliocentric cylindrical coordinate system can be described as

$$r = \frac{a}{1 - \cos\theta}$$

where $a = 1.00\,\mathrm{au}$.

(i) In what range of wavelengths should we be searching for a Dyson sphere created by a type II civilization in a distant galaxy, if its distance from Earth is d and the sphere can operate between temperatures T_1 and T_2 ($T_1 < T_2$). Assume only non-relativistic effects. **(5pt)**

12 Co-orbital satellites (50 Points)

This question applies a method of determining the masses of two approximately co-orbital satellites developed by Dermott and Murray in 1981.
Suppose that two small satellites of masses m_1 and m_2 are approximately co-orbital (moving on very similar orbits) around a large central body of mass M, with $m_1,\ m_2 \ll M$. At any instant, the orbits of the satellites may be approximated as circular Keplerian orbits with radii r_1 and r_2 respectively, although r_1 and r_2 will vary slightly over time due to the mutual gravitational interaction between the satellites.
Figure 4 depicts the shapes of the orbits in the rotating reference frame with zero angular momentum, centred on the central body. We denote by θ the angle $\measuredangle m_1 M m_2$, while R, x_1, and x_2 denote the mean orbital radius and radial deviations of the satellites.
Throughout this problem, write all answers in an inertial reference frame.
Hint: $(1+x)^\alpha \approx 1+\alpha x$ for $\alpha x \ll 1$
First we will determine the value of $\frac{m_1}{m_2}$.

(a) Write down the angular momentum L_i of the satellite with mass m_i when its circular orbit has radius r_i. **(3pt)**

(b) The satiletes total angular momentum $L_1 + L_2$ is conserved. Let $x_1,\ x_2 \ll R$ be the distances as shown in Figure 4. Find a simple relation between the ratios $\frac{m_1}{m_2}$ and $\frac{x_1}{x_2}$. **(8pt)**

Now, we will try and determine the value of $m_1 + m_2$. For next parts, we will use the actual barycenter of the system, which may not be exactly at the center of the planet.

(c) The individual angular momenta of the satellites m_1 and m_2 will vary over time due to their gravitational interactions. Show that the rate of change of the angular momentum of the second satellite is given by

$$\frac{\Delta L_2}{\Delta t} \approx -\frac{Gm_1m_2}{R}h(\theta) \quad \text{where} \quad h(\theta) = \left[\frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} - \sin\theta\right]$$

(18pt)

(d) Show that $s = r_2 - r_1$ satisfies

$$\frac{\Delta s}{\Delta t} \approx -2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta)$$

(8pt)

(e) For the angle $\theta = \measuredangle m_1 M m_2$ as indicated on Figure 4, find an expression for $\dfrac{\Delta\theta}{\Delta t}$ **(5pt)**

(f) Using the results above, find the relation between Δs and $\Delta\theta$. **(2pt)**

(g) After integrating the expression above, we will obtain the result,

$$\overline{x}^2 \approx \frac{4R^2}{3}\frac{m_1+m_2}{M}\left(\frac{1}{\sin\left(\frac{\theta_{\min}}{2}\right)} - 2\cos\theta_{\min} - 3\right)$$

where $\overline{x} = \dfrac{x_1 + x_2}{2}$.

Epimetheus (m_1) and Janus (m_2) are two approximately co-orbital moons of Saturn. Detailed observations of their orbits have been performed by the Voyager 1 and Cassini spacecraft, which found that $R = 150\,000\,\text{km}$, $x_1 = 76\,\text{km}$ and $x_2 = 21\,\text{km}$. The minimum distance between Janus and Epimetheus is 13 000 km. The mass of Saturn is known to be 5.7×10^{26} kg. Estimate the masses of Epimetheus and Janus. **(6pt)**

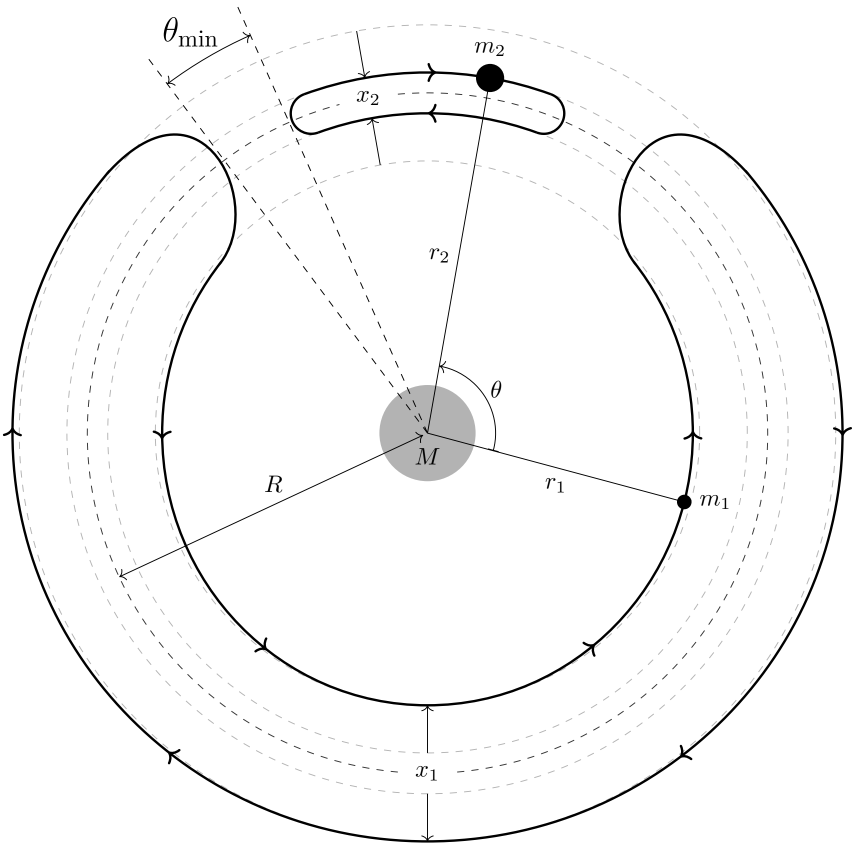


Figure 4: This figure schematically depicts the shapes of the orbits in the rotating reference frame, selected such that in this frame the total angular momentum of the two satellites is zero.

13 Relativistic Beaming (50 Points)

Consider an isotropic light source of frequency f_R in a frame which is fixed to the source (i.e. rest frame). In this rest frame, consider a light ray emitted from the source that makes an angle θ_R with the X-axis. The light source is moving along positive X direction with (relativistic) speed v as measured in the lab frame.

(a) Find an expression for the frequency f_L of this ray in the lab frame, and the cosine of the angle that this ray makes with the X-axis in the lab frame. **(11pt)**
Hint: In relativistic mechanics, energy E and momentum p of a particle between rest and lab frame are related in the following way:

$$\frac{E_L}{c} = \gamma\left(\frac{E_R}{c} + p_{x_R}\frac{v}{c}\right)$$
$$p_{x_L} = \gamma\left(p_{x_R} + \frac{E_R v}{c^2}\right)$$
$$p_{y_L} = p_{y_R}$$
$$p_{z_L} = p_{z_R}$$

where:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

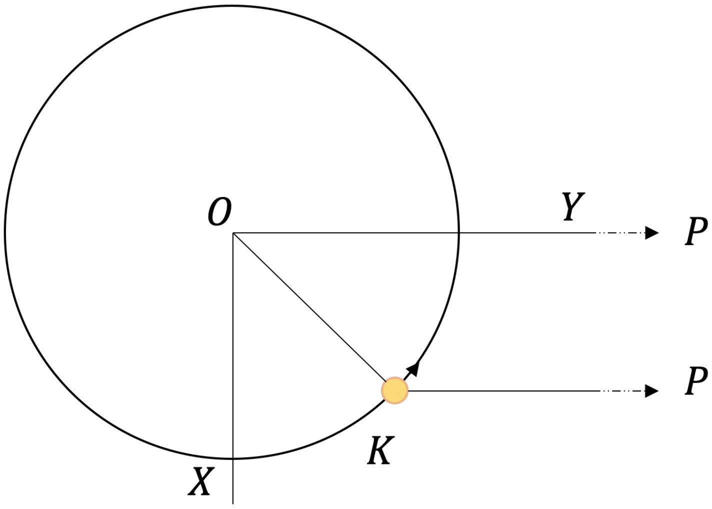
(b) For the following cases:

i) $\theta_R = 0^\circ$
ii) $\theta_R = \cos^{-1}(-v/c)$
iii) $\theta_R = 90^\circ$

iv) $\theta_L = 180°$

draw direction vectors of the beam in XY plane of the rest frame as well as separately in $X'Y'$ plane of the lab frame. **(4 pt)**

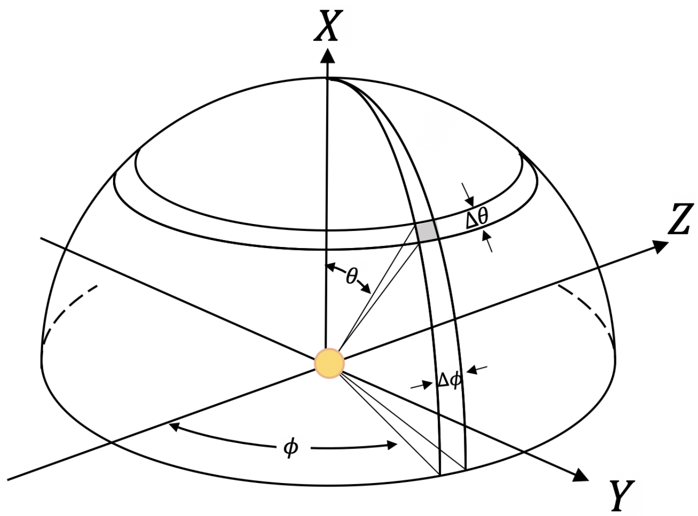
In accretion disks around black holes, the charged particles are orbiting at relativistic speeds and in their rest frames may be considered as isotropic point sources of light. Consider such a particle K in a circular orbit of radius r and angular speed ω around a central object located at O (see figure).



Let us assume that our lab frame is fixed to an observer located at a point P on the OY axis, which is stationary with respect to O. $OP = R \gg r$. Let $t_{L0} = t_{R0} = 0$ correspond to the moment when K is seen crossing the OX axis. As K is moving with relativistic speed, the duration Δt_R measured by an observer in the rest frame of the source K is related to the duration measured in the lab frame Δt_L at P by the expression $\Delta t_L = \gamma \Delta t_R$.

(c) Derive an expression for f_L as a function of t_L $(t_L > R/c)$? **(7pt)**

Let us consider a fraction of the light from the source that is emitted in an infinitesimal solid angle $\Delta\Omega_R = -\Delta(\cos\theta_R)\cdot\Delta\phi$ in the direction making an angle θ_R with respect to the X axis in the rest frame, as it is shown on the figure below.



(d) Show that, as measured in the lab frame

$$\Delta\Omega_L = \frac{\Delta\Omega_R}{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2}$$ **(10pt)**

(e) If the intrinsic luminosity of the light source is L, what is the energy flux F_L observed by the observer at point P at the moment t_L $(t_L > R/c)$? **(15pt)**
Hint: In the rest frame of the source, you may assume N_R number of photons are directed within the solid angle $\Delta\Omega_R$ during the time interval Δt_R.

(f) Charged particles in the relativistic beam shot from the supermassive black hole at the centre of the galaxy M87 have speeds up to $0.95c$. What would be the maximum and minimum amplification factor for the energy flux for a relativistic beam from M87? **(3pt)**

1 Solution: Planck's units (10 points)

(a) From dimensional analysis

$$c \equiv \left[L^1 M^0 T^{-1}\right] \qquad \mathbf{1}$$
$$G \equiv \left[L^3 M^{-1} T^{-2}\right] \qquad \mathbf{1}$$
$$\hbar \equiv \left[L^2 M^1 T^{-1}\right] \qquad \mathbf{1}$$
$$k_B \equiv \left[L^2 M^1 T^{-2} \Theta^{-1}\right] \qquad \mathbf{1}$$
$$G\hbar \equiv \left[L^5 M^0 T^{-3}\right]$$
$$\frac{G\hbar}{c^3} \equiv \left[L^2 M^0 T^0\right]$$
$$\therefore L_p = \sqrt{\frac{G\hbar}{c^3}} \qquad \mathbf{1.5}$$
$$\frac{\hbar c}{G} \equiv \left[L^0 M^2 T^0\right]$$
$$\therefore m_p = \sqrt{\frac{\hbar c}{G}} \qquad \mathbf{1.5}$$
$$\frac{L_p}{c} \equiv \left[L^0 M^0 T^1\right]$$
$$\therefore t_p = \sqrt{\frac{\hbar G}{c^5}} \qquad \mathbf{1.0}$$
$$T_p = \frac{L_p^2 m_p}{k_B t_p^2} = \frac{G\hbar}{k_B c^3} \times \frac{c^5}{\hbar G} \times \sqrt{\frac{\hbar c}{G}}$$
$$\therefore T_p = \sqrt{\frac{\hbar c^5}{G k_B^2}} \qquad \mathbf{1.0}$$
$$4\pi\epsilon_0 \equiv \left[L^{-3} M^{-1} T^2 Q^2\right]$$
$$\hbar c \equiv \left[L^3 M^1 T^{-2}\right]$$
$$\therefore q_p = \sqrt{4\pi\epsilon_0 \hbar c} \qquad \mathbf{1.0}$$

2 Solution: Circumbinary planet (10 points)

Equating gravitational force to centrifugal force, we get:

$$\frac{4\pi^2}{T_b^2} a_b = \frac{G2M_\odot}{4a_b^2}$$
$$\frac{T_b^2}{a_b^3} = 2\frac{4\pi^2}{GM_\odot}$$
$$\implies a_b = 2\,\text{au}. \qquad \mathbf{2.0}$$

The period of planet can be calculated from the 3rd Kepler's law of motion

$$\frac{T_p^2}{a_p^3} = \frac{4\pi^2}{GM_\Sigma} = \frac{4\pi^2}{4GM_\odot}$$
$$\implies T_p = 44.7\,\text{years} \qquad \mathbf{2.0}$$
$$\text{Now, } \frac{1}{T_s} = \frac{1}{T_b} - \frac{1}{T_p}$$
$$\therefore T_s = \frac{T_b T_p}{T_p - T_b} = 4.4\,\text{years} \qquad \mathbf{2.0}$$

One star always covers exactly the half of the planet's surface. In the best case, the stars are located at the maximal angular distance from each other. So the fraction of illuminated surface is simply given by

$$n = \frac{1}{2} + \frac{\theta}{2\pi} = \frac{1}{2} + \frac{2arctan\left(\frac{a_b}{a_p}\right)}{2\pi} = 0.53. \qquad \textbf{4.0}$$

3 Solution: Expanding ring nebula (10 points)

(a) The distance traveled by the gas particles of inner and outer radius can be calculated as follows:

$$S_{out} = d_\oplus(\theta'_{\text{out}} - \theta_{\text{out}}) = 100pc \cdot (8.0' - 4.0') = 24 \times 10^3 \,\text{au}$$
$$S_{in} = d_\oplus(\theta'_{\text{in}} - \theta_{\text{in}}) = 100pc \cdot (7.0' - 3.5') = 21 \times 10^3 \,\text{au} \qquad \textbf{2.0}$$
$$\therefore v_{max} = \frac{S_{out}}{t_0} = 57.0 \,\text{km/s}$$
$$v_{min} = \frac{S_{in}}{t_0} = 50.0 \,\text{km/s} \qquad \textbf{2.0}$$

(b) At the inner edge, escape velocity from the white dwarf can be maximal ($1.44M_\odot$- Chandrasekhar limit)

$$v_{esc} = \sqrt{\frac{2G \cdot 1.44M_\odot}{S_{in}}} \approx 0.35 \,\text{km/s} \qquad \textbf{2.0}$$

Since $v_{esc} \ll v_{min}$, the no-gravity scenario can be applied. YES **1.0**
(c)

$$\phi = 8' - 7.5' = 0.5' = 30'' \geq \frac{1.22\lambda}{D}$$
$$\therefore D \geq \frac{1.22\lambda}{\phi} = \frac{1.22 \times 5 \times 10^{-7} \times 206265}{30} = 4.19 \,\text{mm} \qquad \textbf{2.0}$$

Hence YES **1.0**

4 Solution: Journey Between Galaxies (10 points)

(a) By the Hubble's Law, the rate of recession is

$$v(t) = \frac{\Delta d}{\Delta t} = Hd(t) \qquad \textbf{1.0}$$

Thus, by referring to the given relation,

$$d(t) = Ce^{Ht}$$

at time $t = 0$, $d(0) = d_0 = C$. Thus, the distance at a time t:

$$d(t) = d_0 e^{Ht} \qquad \textbf{1.0}$$

(b) It is easier to solve this problem, if we consider coordinate system that changes its scale in such a way that distance between earth and our destination does not change (called co-moving frame). Let $v(t)$ be the velocity in this frame at time t and let $l(t)$ be the total distance traveled until the moment t.

$$v(t)d(t) = v_0 d_0$$
$$v(t) = v_0 \frac{d_0}{d(t)} = v_0 \frac{d_0}{d_0 e^{Ht}}$$
$$v(t) = \frac{\Delta l}{\Delta t} = -v_0 e^{-Ht} \qquad \textbf{3.0}$$

$$l(t) = -\frac{v_0}{-H} e^{-Ht} + C$$

for specific cases:

$$l(t_0) = \frac{v_0}{H} e^{-Ht_0} + C$$

$$l(0) = \frac{v_0}{H} + C$$

subtracting yields:

$$l(0) = \frac{v_0}{H} - \frac{v_0}{H}e^{-Ht_0}$$

thus:

$$\therefore l_{\text{Total}} = d_0 = -\frac{v_0}{H}(e^{-Ht_0} - 1) \quad \textbf{2.0}$$

Rearranging,

$$t_0 = -\frac{1}{H} ln\left(1 - \frac{Hd_0}{v_0}\right) \quad \textbf{1.0}$$

Thus, condition for reaching the planet at all:

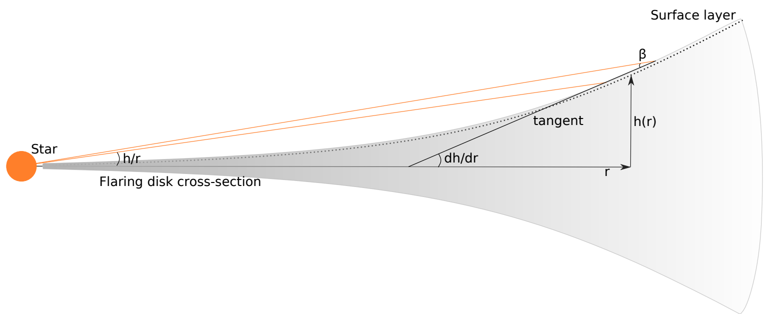
$$1 - \frac{Hd_0}{v_0} > 0$$

$$v_0 > Hd_0 = 70\,\text{km/s} \quad \textbf{1.0}$$

(However at this speed it will take eternity to reach it)
For $v_0 = 1000\,\text{km/s}$ we can reach the planet, in:

$$t_0 \approx 5.3 \times 10^{16}\,\text{s} = 17\,\text{Gyr} \quad \textbf{1.0}$$

5 Solution: Flaring protoplanetary disk (10 points)



(a) The angle between the light beam and the horizontal plane is

$$\theta_l \approx \tan\theta_l = \frac{h(r)}{r} \quad \textbf{1.0}$$

Additionally, a tangent is inclined to the disc plane with an angle of

$$\theta_T \approx \tan\theta_T = \frac{\Delta h(r)}{\Delta r} \quad \textbf{2.0}$$

An exterior angle of a triangle is equal to the sum of its two interior opposite angles. Thus,

$$\beta = \theta_T - \theta_l = \frac{\Delta h(r)}{\Delta r} - \frac{h(r)}{r} \quad \textbf{1.0}$$

(b) The flux from of stellar radiation at distance r from the star is $E_s = \frac{L_s}{4\pi r^2}$, where L_s is the luminosity of the star. However, the irradiation flux is the normal projection of this flux onto the infinitesimal small surface A of the disk.

$$Q_+ = E_s A \sin\beta \approx \frac{L_s A\beta}{4\pi r^2} \quad \textbf{1.5}$$

From the Stefan-Boltzmann law, the cooling rate is give by

$$Q_- = A\sigma T_D^4. \quad \textbf{0.5}$$

In the thermal equilibrium

$$Q_+ = Q_- \implies T_D = \left(\frac{L_s\beta}{4\pi\sigma r^2}\right)^{\frac{1}{4}} \qquad \textbf{1.0}$$

(c) The condition for the isothermal layer can be solved by applying

$$\begin{aligned}\beta &= \frac{\Delta h(r)}{\Delta r} - \frac{h(r)}{r} = \frac{a(r+\Delta r)^b - ar^b}{\Delta r} - \frac{ar^b}{r}\\ &= ar^b\left(\frac{(1+\frac{\Delta r}{r})^b - 1}{\Delta r} - \frac{1}{r}\right) = ar^b\left(\frac{b\Delta r}{r\Delta r} - \frac{1}{r}\right)\\ &= a(b-1)r^{b-1} \propto r^2\\ \implies b &= 3\end{aligned} \qquad \textbf{2.0}$$

Finally, one can determine the second constant by

$$\begin{aligned}\beta &= a(b-1)r^{b-1} = 2ar^2\\ \therefore T_{SL}^4 &= \left(\frac{aL_s}{2\pi\sigma}\right)\\ a &= \left(\frac{2\pi\sigma T_{SL}^4}{L_s}\right)\end{aligned} \qquad \textbf{1.0}$$

6 Solution: Photometry of Binary stars (20 Points)

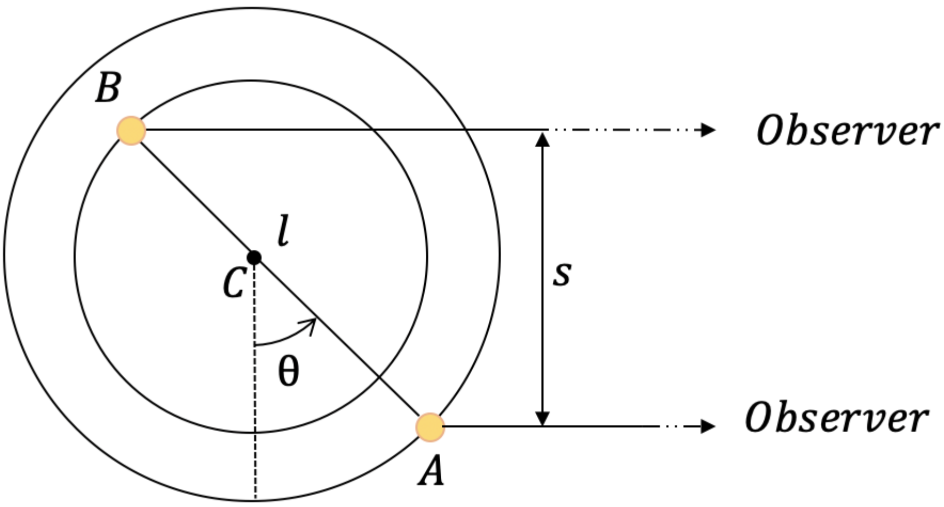
(a) From the Wien's displacement law the temperature of a black body is related to a peak wavelength of emission by the relation:

$$\lambda_{max}T = b \qquad \textbf{1.0}$$
$$T_A = 5978K \qquad \textbf{0.5}$$
$$T_B = 4830K \qquad \textbf{0.5}$$

(b) Due to the diffraction minimal angular separation between two celestial objects to distinguish them:

$$\delta = 1.22\frac{\lambda}{D} = 4.44 \times 10^{-8}\,\text{rad} = 9.16\,\text{mas} \qquad \textbf{1.0}$$

Now let us derive the time dependence of the angular separation between two stars as seen from the earth



As it is seen from the figure, distance $s = l\cos\theta$, where l is the distance between stars and θ is the angle of rotation of the stars about their center of mass after the stars are seen with maximum angular separation. The angular separation seen by an observer on the earth:

$$\gamma = \frac{s}{d} = \frac{l}{d}\cos\theta \qquad \textbf{1.0}$$

The 38 days, during which objects are seen as a one object the system rotates by the angle $\beta = 2\pi\frac{38}{100}$. **1.0**

Thus, the corresponding starting angle θ' at which stars are no longer distinguishable:

$$\theta' = \frac{\pi}{2} - \frac{\beta}{4} = \frac{\pi}{2} - \frac{19\pi}{100} = \frac{31\pi}{100}$$ **1.5**

and this is the moment, when stars stop to be distinguishable. Thus, corresponding angular separation on the sky is δ:

$$\delta = \frac{l}{d}\cos\theta'$$

From which one obtains

$$l = \frac{\delta d}{\cos\theta'} = 1.45\,\mathrm{au}$$ **1.5**

(c) From the Kepler's 3rd law:

$$\begin{aligned} T^2 &= \frac{4\pi^2 l^3}{G(M_A + M_B)} \\ M_A + M_B &= \frac{4\pi^2 l^3}{GT^2} \\ &= 40.7M_\odot \end{aligned}$$ **2.0**

(d) From given data for case 1:

$$\begin{aligned} (U-V)_1 &= (U-B)_1 + (B-V)_1 = 0.3 \\ U_1 &= U_{0_1} + a_U d = 6.51 \\ V_1 &= U_1 - (U-V)_1 = 6.21 \\ m_{Bol_1} &= V_1 + BC_1 = 6.31 \end{aligned}$$ **1.5**

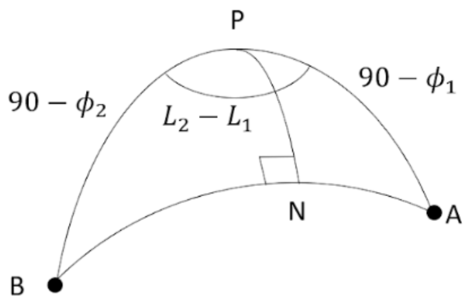
Similarly for second configuration:

$$\begin{aligned} (U-V)_2 &= (U-B)_2 + (B-V)_2 = 0.37 \\ U_2 &= U_{0_2} + a_U d = 6.98 \\ V_2 &= U_2 - (U-V)_2 = 6.61 \\ m_{Bol_2} &= V_2 + BC_2 = 6.79 \end{aligned}$$ **1.0**

The difference in bolometric magnitudes:

$$
\begin{aligned}
m_{bol_2} - m_{bol_1} &= -2.5\log\left(\frac{L_2}{L_1}\right) \\
&= -2.5\log\left(\frac{\pi(R_A^2 - R_B^2)\sigma T_A^4 + \pi R_B^2\sigma T_B^4}{\pi R_A^2\sigma T_A^4 + \pi R_B^2 \sigma T_B^4}\right) && \textbf{1.5} \\
\therefore 6.79 - 6.31 &= -2.5\log\left(\frac{(R_A^2 - R_B^2)T_A^4 + R_B^2 T_B^4}{R_A^2 T_A^4 + R_B^2 T_B^4}\right) \\
&= -2.5\log\left(\frac{(R_A^2/R_B^2 - 1)T_A^4 + T_B^4}{T_A^4 R_A^2/R_B^2 + T_B^4}\right) \\
\therefore 0.48 &= -2.5\log\left[\frac{\left(\frac{R_A^2}{R_B^2} - 1\right) + \frac{T_B^4}{T_A^4}}{\frac{T_B^4}{T_A^4} + \frac{R_A^2}{R_B^2}}\right] && \textbf{1.0} \\
\therefore 10^{\frac{-0.48}{2.5}} &= \frac{\frac{R_A^2}{R_B^2} - 1 + \frac{T_B^4}{T_A^4}}{\frac{T_B^4}{T_A^4} + \frac{R_A^2}{R_B^2}} = 10^{-0.19} = 0.65 && \textbf{1.0} \\
\frac{R_A^2}{R_B^2} - 1 + \frac{T_B^4}{T_A^4} &= 0.65\frac{T_B^4}{T_A^4} + 0.65\frac{R_A^2}{R_B^2} \\
0.35\frac{R_A^2}{R_B^2} &= 1 - 0.35\frac{T_B^4}{T_A^4} \\
&= 1 - 0.35 \times 0.482 = 0.83 \\
\therefore \frac{R_A}{R_B} &= \sqrt{\frac{0.83}{0.35}} = 1.54 && \textbf{1.0} \\
\frac{M_A}{M_B} &= \frac{\rho_A R_A^3}{\rho_B R_B^3} = 1.54^3 \cdot 0.7 = 2.56 && \textbf{1.0} \\
\text{But, } M_A + M_B &= 40.7 M_\odot && \textbf{1.0} \\
\therefore M_B &= \frac{40.7 M_\odot}{3.56} = 11.4 M_\odot && \textbf{1.0} \\
M_B &= 29.3 M_\odot
\end{aligned}
$$

7 Solution: Georgia to Georgia (20 points)



In the diagram, N is the northernmost point of the path $\widehat{AB}$. Triangle APB, triangle APN and triangle BPN are all spherical triangles. We shall use the convention that North and West are positive, and South and East are negative. We notice,

$$\begin{aligned}
\widehat{\mathrm{PN}} &= -\delta_F = 30°4' \\
\widehat{\mathrm{PA}} &= 90° - \phi_A = 48°17' \\
\widehat{\mathrm{PB}} &= 90° - \phi_B \\
\sphericalangle PNA &= \sphericalangle PNB = 90° \\
\sphericalangle BPA &= \lambda_B - \lambda_A
\end{aligned}$$ **5.0**

$$\begin{aligned}
x = \widehat{\mathrm{AB}} &= vt = 4375 \times 2\frac{17}{60} = 9990\,\mathrm{km} \\
\therefore x &= \frac{9900}{R_\oplus} = 1.5663\,\mathrm{rad} \\
&= 89.74° = 89°44'
\end{aligned}$$ **2.0**

In $\triangle APN$, using sine rule,

$$\begin{aligned}
\frac{\sin\sphericalangle A}{\sin\widehat{\mathrm{PN}}} &= \frac{\sin\sphericalangle N}{\sin\widehat{\mathrm{PA}}} \\
\therefore \sin\sphericalangle A &= \frac{\sin 90° \sin(30°4')}{\cos\phi_A} = \frac{\sin(30°4')}{\cos(41°43')} = 0.67119 \\
\therefore \sphericalangle A &= 0.7358\,\mathrm{rad} = 42.16° = 42°10'
\end{aligned}$$ **3.0**

(Recognise that to have a northern point on the path, it must be the acute solution)

In $\triangle PBA$, using cosine rule,

$$\begin{aligned}
\cos\widehat{\mathrm{PB}} &= \cos\widehat{\mathrm{PA}}\cos\widehat{\mathrm{AB}} + \sin\widehat{\mathrm{PA}}\sin\widehat{\mathrm{AB}}\cos\sphericalangle A \\
\sin\phi_B &= \sin\phi_A\cos x + \cos\phi_A\sin x\cos A \\
&= \sin(41°43')\cos(89°44') + \cos(41°43')\sin(89°44')\cos(42°10') \\
\sin\phi_B &= 0.5563 \\
\therefore \phi_B &= 0.5900\,\mathrm{rad} = 33.80° = 33°48'
\end{aligned}$$ **4.0**

(Only one solution in the valid values of latitude)

Again using cosine rule,

$$\begin{aligned}
\cos\widehat{\mathrm{AB}} &= \cos\widehat{\mathrm{PA}}\cos\widehat{\mathrm{PB}} + \sin\widehat{\mathrm{PA}}\sin\widehat{\mathrm{PB}}\cos\sphericalangle P \\
\therefore \cos\sphericalangle P &= \frac{\cos x - \sin\phi_A\sin\phi_B}{\cos\phi_A\cos\phi_B} \\
&= \frac{\cos(89°44') - \sin(41°43')\sin(33°48')}{\cos(41°43')\cos(33°48')} \\
\cos\sphericalangle P &= -0.5891 \\
\therefore \sphericalangle P &= 2.201\,\mathrm{rad} = 126.13° = 126°8' \\
\lambda_B &= \sphericalangle P + \lambda_A = 126°8' - 41°48' \\
\lambda_B &= 1.472\,\mathrm{rad} = 84.33° = 84°20'
\end{aligned}$$ **5.0**

(The other solution of $\lambda_B - \lambda_A = -126.13°$ gives $\lambda_B = -167.93°$ which is clearly not valid)

Therefore, the coordinates of Atlanta are

$$(33°48'\mathrm{N}, 84°20'\mathrm{W})$$ **1.0**

Alternative method to find L_2
Using the spherical sine rule in triangle PBA,

$$\frac{\sin A}{\sin(90° - \phi_2)} = \frac{\sin(L_2 - L_1)}{\sin x}$$

$$\therefore \sin(L_2 - L_1) = \frac{\sin x \sin A}{\cos \phi_2}$$

$$\therefore \sin(L_2 - L_1) = \frac{\sin(89°51') \sin(42°10')}{\cos(33°43')} = 0.80689$$

$$\therefore L_2 - L_1 = 126.21° \quad (= 126°12' = 2.20\text{rad})$$
$$\therefore L_2 = 126°12' + L_1 = 126°12' + (-41°48') = 84.41° \quad (= 84°24' = 1.47\text{rad})$$

(The other solution of $L_2 - L_1 = 53.79°$ gives $L_2 = 11.99°$ which is clearly not valid)

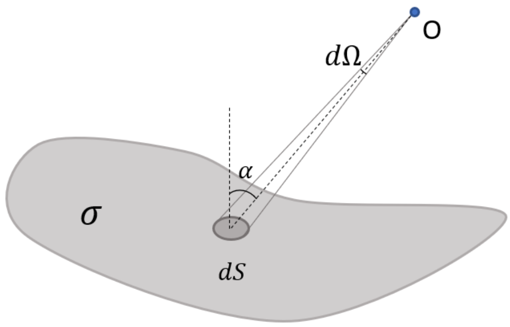
Accept all other valid solutions (such as the four parts rule or using Napier's rules) that give $\phi_2 = 33.71°$ and $L_2 = 84.41°$. Students may do both methods for calculating L_2 to confirm which solution is common to both as a quicker check than trying to do the reverse calculation with a given pair of co-ordinates.

8 Solution: Saturn's Rings (20 Points)

(a) Consider small surface element (ΔS) of the disk. The gravitational field (ΔE) generated by this surface element at at a point O located at a distance a from ΔS (see the figure below):

$$\Delta E = G\frac{\sigma \Delta S}{a^2}$$

where σ is a surface density of the mass.



where α is the angle between surface normal and the vector $\vec{r}$ and $\Delta\Omega$ is infinitesimal solid angle corresponding to the area ΔS.
For a ring, the surface density will be given by $\sigma = M/\pi(R^2 - r^2)$.
As the horizontal component will get cancelled due to symmetry considerations, we only consider the normal component:

$$\Delta E_\perp = \Delta E cos(\alpha) = \frac{G\sigma}{a^2}\Delta S cos(\alpha) = \frac{G\sigma}{a^2}\Delta S_\perp = G\sigma\Delta\Omega \qquad \textbf{5.0}$$

By Summation, we can obtain total normal field of the surface:

$$E_\perp = G\sigma\Omega \qquad \textbf{1.0}$$

here Ω is the total solid angle subtended by the disk. In our case, when we observe disk from the point of view of the test particle:

$$\Omega = 2\pi(1 - \cos\theta_{out}) - 2\pi(1 - \cos\theta_{in})$$
$$= 2\pi\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right) - 2\pi\left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right)$$
$$\Omega = 2\pi x\left(\frac{1}{\sqrt{x^2 + r^2}} - \frac{1}{\sqrt{x^2 + R^2}}\right) \qquad \textbf{3.0}$$
$$\therefore F_\perp = G\sigma\Omega m$$
$$F_\perp = \frac{2GMmx}{R^2 - r^2}\left(\frac{1}{\sqrt{x^2 + r^2}} - \frac{1}{\sqrt{x^2 + R^2}}\right) \qquad \textbf{1.0}$$

(b) In small displacement approximation:

$$F_{\perp} \approx \frac{2GMmx}{R^2 - r^2}\left(\frac{1}{r} - \frac{1}{R}\right) = \left(\frac{2GMm}{Rr(R+r)}\right)x = kx \qquad \textbf{3.0}$$

when the distance between the disk and the point mass is x, the distance of the center of mass of the system from the center of the disk will be:

$$x_1 = \frac{mx}{m+M} \qquad \textbf{2.5}$$

If we choose the center of mass of the system as an origin of our coordinate system, then equation of motion of the disk will have a form:

$$Ma_1 = -kx$$

$$Ma_1 + k\frac{m+M}{m}x_1 = 0 \qquad \textbf{2.5}$$

For this simple harmonic oscillator, the period of small oscillations will be:

$$T = 2\pi\sqrt{\frac{Rr(R+r)}{2G(M+m)}} \qquad \textbf{2.0}$$

9 Solution: Solar Retrograde Motion on Mercury (20 points)

From Kepler's 3rd Law, $T^2 = a^3$ in units of Earth years and au respectively.

$$\therefore T_{\text{Mercury}} = \sqrt{(0.387)^3} = 0.240\,75\,\text{yr} = 87.934\,\text{days} \qquad \textbf{2.0}$$

We know that $\omega_{\text{orb}} = v/r$ so will vary throughout the orbit (since r varies within an ellipse) whilst ω_{rot} is constant.

$$\omega_{\text{rot}} = \frac{2\pi}{T_{\text{rot}}} = \frac{2\pi}{\frac{2}{3}\times 87.934 \times 86400} = 1.24\times 10^{-6}\,\text{rad/s} \qquad \textbf{2.0}$$

For retrograde motion, we need $\omega_{\text{orb}} \geq \omega_{\text{rot}}$.
Using the vis-viva equation for the critical value of r for when $\omega_{\text{rot}} = \omega_{\text{orb}}$,

$$\omega_{\text{orb}}^2 = \frac{v^2}{r^2} = \frac{GM_\odot}{r^2}\left(\frac{2}{r} - \frac{1}{a}\right) = \omega_{\text{rot}}^2$$

$$\frac{2GM_\odot}{r^3} - \frac{GM_\odot}{ar^2} = \omega_{\text{rot}}^2$$

$$\therefore \frac{\omega_{\text{rot}}^2}{GM_\odot}r^3 + \frac{1}{a}r - 2 = 0$$

$$(1.160\times 10^{-32})r^3 + (1.725\times 10^{-11})r - 2 = 0 \quad \text{(if } r \text{ is in meters)}$$

$$38.99r^3 + 2.584r - 2 = 0 \quad \text{(if } r \text{ is in au)} \qquad \textbf{4.0}$$

Solving the cubic equation (by any valid method) gives only one non-imaginary root
One possible way would be to use iterations for this polynomial $f(r)$, using perihelion distance as the starting guess.

r	$f(r)$
0.3073	-0.07482
0.3100	-0.03747
0.3120	-0.00967
0.3140	0.01841
0.3130	0.00433
0.3127	0.00012

$$\therefore r = 0.3127\,\text{au} = 4.684\times 10^{10}\,\text{m} \qquad \textbf{4.0}$$

From knowledge of ellipses, if E is the eccentric anomaly

$$r = a(1 - e\cos E)$$
$$\therefore E = \cos^{-1}\left[\frac{1}{e}\left(1 - \frac{r}{a}\right)\right] = \cos^{-1}\left[\frac{1}{0.206}\left(1 - \frac{0.3127}{0.387}\right)\right]$$
$$E = 0.3706\,\text{rad} = 21.23° = 21°14' \qquad \textbf{3.0}$$

Using Kepler's Equation we can find the mean anomaly, M

$$M = E - e\sin E = 0.3706 - 0.206 \times \sin 0.3706$$
$$= 0.2960\,\text{rad} = 16.96° = 16°58' \qquad \textbf{2.0}$$

This is relative to the perihelion, so symmetry demands that the total time the sun is in retrograde corresponds to

$$\Delta M = 2M$$
$$\therefore T_{\odot,retro} = T_{\text{Mercury}}\frac{\Delta M}{2\pi} = 87.934 \times \frac{2 \times 0.296}{2\pi}$$
$$T_{\odot,retro} = \boxed{8.28\,\text{days}} \qquad \textbf{3.0}$$

[Accept alternative methods making use of the true anomaly e.g. with the relation $\cos E = \frac{e+\cos\nu}{1+e\cos\nu}$.
]

10 Solution: Accretion (20 Points)

(a) Consider a particle at a distance r from the center of the compact object. In case of maximal luminosity, the gravitational force must be balanced by the radiation pressure.

$$F_G = G\frac{m_H M}{r^2} \qquad \textbf{0.5}$$
$$F_R = \frac{\sigma_e L_E}{4\pi c r^2} \qquad \textbf{0.5}$$
$$\text{But, } F_G = F_R \qquad \textbf{1.0}$$
$$\therefore L_E = \frac{4\pi G m_H M c}{\sigma_e} \qquad \textbf{1.0}$$

(b) The lower bound of its mass given by:

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2$$
$$\approx 6.65 \times 10^{-29}\,\text{m}^2 \qquad \textbf{1.5}$$
$$M = \frac{\sigma_e L_\odot}{4\pi G m_H c}$$
$$= 6.04 \times 10^{25}\,\text{kg}$$
$$= 3.04 \times 10^{-5} M_\odot \qquad \textbf{2.5}$$

(c) Gravitational energy of atoms of total mass Δm:

$$\Delta E = \frac{GM\Delta m}{R}$$
$$L_{\text{acc}} = \frac{\Delta E}{\Delta t}$$
$$\therefore L_{\text{acc}} = \frac{GM}{R}\frac{\Delta m}{\Delta t} = \frac{GM}{R}\dot{M} \qquad \textbf{2.0}$$

(d) For maximum $\dot{M}$:

$$L_{\text{acc}} = L_E$$
$$\frac{GM}{R}\dot{M} = \frac{4\pi G m_H M c}{\sigma_e}$$
$$\dot{M} = \left(\frac{4\pi m_H c}{\sigma_e}\right) R \qquad \textbf{2.0}$$

(e) For given value of R_*:

$$\dot{M} = \left(\frac{4\pi m_H c}{\sigma_e}\right) R_*$$
$$\dot{M} = 1.14 \times 10^{21}\,\text{kg/s}$$
$$= 3.59 \times 10^{28}\,\text{kg/yr}$$
$$\dot{M} = 0.018\,\text{M}_\odot/\text{yr} \quad \textbf{2.0}$$

(f) The total angular momentum and total mass must be conserved.

$$L = M_1 r_1^2 \Omega + M_2 r_2^2 \Omega \quad \textbf{1.0}$$
$$\text{Let } M_T = M_1 + M_2$$
$$\text{and } a = r_1 + r_2$$
$$r_1 = \frac{M_2 a}{M_T} \quad \text{and} \quad r_2 = \frac{M_1 a}{M_T} \quad \textbf{1.0}$$
$$\therefore L = \left[M_1\left(\frac{M_2 a}{M_T}\right)^2 + M_2\left(\frac{M_1 a}{M_T}\right)^2\right]\Omega$$
$$= \left(M_1 M_2^2 + M_2 M_1^2\right)\frac{a^2\Omega}{M_T^2}$$
$$\therefore L = \frac{a^2\Omega M_1 M_2}{M_T} == \frac{a^2\Omega M_1(M_T - M_1)}{M_T} \quad \textbf{1.0}$$
$$\therefore a = \left(\sqrt{\frac{LM_T}{\Omega}}\right) M_1^{-0.5}(M_T - M_1)^{-0.5}$$
$$a = KM_1^{-0.5}(M_T - M_1)^{-0.5} \quad \textbf{1.0}$$

Now let us say mass of the compact object **increases** by small fraction ΔM_1. Let the corresponding **increases** in the separation be Δa.

$$a + \Delta a = K(M_1 + \Delta M_1)^{-0.5}(M_T - M_1 - \Delta M_1)^{-0.5}$$
$$= KM_1^{-0.5}(M_T - M_1)^{-0.5}\left(1 + \frac{\Delta M_1}{M_1}\right)^{-0.5}\left(1 - \frac{\Delta M_1}{M_T - M_1}\right)^{-0.5}$$
$$a + \Delta a = a\left(1 + \frac{\Delta M_1}{M_1}\right)^{-0.5}\left(1 - \frac{\Delta M_1}{M_2}\right)^{-0.5} \quad \textbf{1.0}$$
$$\therefore 1 + \frac{\Delta a}{a} \approx \left(1 - \frac{\Delta M_1}{2M_1}\right)\left(1 + \frac{\Delta M_1}{2M_2}\right)$$
$$\approx 1 - \frac{\Delta M_1}{2M_1} + \frac{\Delta M_1}{2M_2}$$
$$\therefore \frac{\Delta a}{a} = \frac{\Delta M_1}{2}\left(\frac{M_1 - M_2}{M_1 M_2}\right) \quad \textbf{1.0}$$

Thus, Δa is positive (separation increases) when $M_1 > M_2$ and the separation decreased when $M_1 < M_2$. **1.0**

11 Solution: Dyson Sphere (50 Points)

(a) Heat absorbed must be fully emitted to maintain the thermal equilibrium

$$kL_\odot = 4\pi R^2 \epsilon\sigma T_{\text{eq}}^4 \quad \textbf{2.0}$$
$$\therefore T_{\text{eq}} = \sqrt[4]{\frac{kL_\odot}{4\pi R^2 \epsilon\sigma}} \quad \textbf{1.0}$$

(b) In this part, we try minimize the radius at the expanse of reaching the highest operational temperature for panels. Again

$$kL_\odot = 4\pi R^2 \epsilon\sigma T_{\max}^4$$

$$R = \sqrt{\frac{kL_\odot}{4\pi\epsilon\sigma T_{\max}^4}} \quad \textbf{1.0}$$

$$\approx 1\times 10^{11}\,\text{m} = 0.665\,\text{au} \quad \textbf{2.0}$$

Clearly, this distance is approximately 1.5 times smaller than Earth-Sun separation. Hence, Earth will stay **OUT**side the sphere. **1.0**

(c) Power transmitted into usable energy

$$P = \eta L_\odot$$

$$P = 7.65\times 10^{25}\,\text{W} \quad \textbf{2.0}$$

(d) Energy harnessed in 1 second will be enough for

$$\tau = \frac{7.65\times 10^{25}\,\text{W}\times 1\,\text{s}}{17\times 10^{12}\,\text{W}} = 4.5\times 10^{12}\,\text{s} \approx 143\,000\,\text{yr} \quad \textbf{2.0}$$

(e) New equilibrium temperature may be found as

$$kL_\odot \frac{\pi R_\oplus^2}{4\pi a_\oplus^2} = 4\pi R_\oplus^2 \sigma T_{\text{new}}^4 \quad \textbf{2.0}$$

$$T_{\text{new}} = \sqrt[4]{\frac{kL_\odot}{16\pi\sigma a_\oplus^2}}$$

$$T_{\text{new}} \approx 206\,\text{K} \quad \textbf{2.0}$$

$$\Delta T = 288 - 206 = 82\,\text{K} \quad \textbf{1.0}$$

(f) By Kepler's Third Law

$$T^2 = \frac{4\pi^2}{GM_\odot} R^3$$

$$\therefore T = (0.665)^{1.5}\text{yr}$$

$$T \approx 0.542\,\text{yr} = 198\,\text{days} \quad \textbf{3.0}$$

(g) We compare the two forces for given area A of the solar panel

$$F_{\text{RP}} = \frac{I_{inc}A}{c} = \frac{L_\odot A}{4\pi R^2 c} \quad \textbf{2.0}$$

$$F_{\text{Grav}} = \frac{GM_\odot A\rho}{R^2} \quad \textbf{1.0}$$

$$\alpha = \frac{F_{\text{RP}}}{F_{\text{Grav}}} = \frac{L_\odot}{4\pi c GM_\odot \rho}$$

$$\alpha \approx 7.65\times 10^{-4} \quad \textbf{2.0}$$

At the first sight, this might seem like a negligible effect, but let us look at the change of the orbital period

$$(1-\alpha)F_{\text{Grav}} = m\omega_1^2 R \quad \textbf{3.0}$$

The radius should be the same, because our aim is to minimize the radius.

$$\omega_1 = \sqrt{\frac{GM_\odot(1-\alpha)}{R^3}}$$

$$T_1 = 2\pi\sqrt{\frac{R^3}{GM_\odot}}(1-\alpha)^{-0.5} \quad \textbf{1.0}$$

$$\therefore \frac{T_1}{T} = (1-\alpha)^{-0.5} \approx 1 + 0.5\alpha$$

$$\Delta T = 0.5\alpha T$$

$$\Delta T \approx 1.8\,\text{h} \quad \textbf{3.0}$$

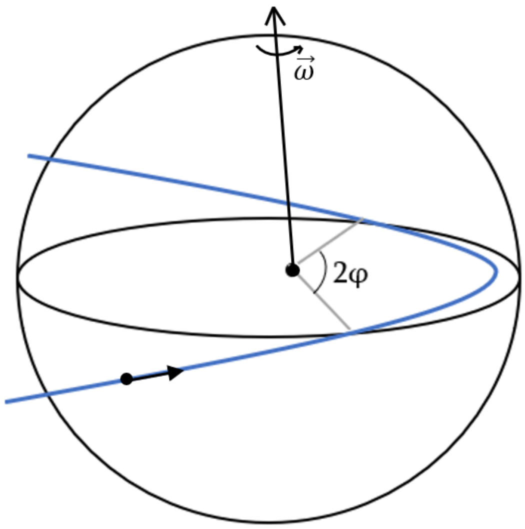


Figure 1: Transit of a asteroid through the sphere.

Therefore, **YES** this additional force has an easily detectable effect. **1.0**

(h) First, we find the angle between entrance and exit points

$$r = \frac{a}{1-\cos\theta}$$
$$R - R\cos\theta = a$$
$$\cos\theta = \frac{R-a}{R} = \frac{0.665-1}{0.665} \qquad \mathbf{1.0}$$
$$\cos\theta = -0.5046 \qquad \mathbf{1.0}$$
$$\theta_1 = 2.099\,\mathrm{rad} = 120.2^\circ = 120^\circ 14'$$
$$\theta_2 = 4.185\,\mathrm{rad} = 239.8^\circ = 239^\circ 46'$$
$$2\phi = 2.086\,\mathrm{rad} = 119.5^\circ = 119^\circ 32' \qquad \mathbf{3.0}$$

We know the that asteroid will stay inside the sphere for about $\tau_0 = 37$ days. During this time, sphere will rotate with an angle

$$\Delta\gamma = \omega\tau_0 = \frac{2\pi\tau_0}{T}$$
$$\Delta\gamma \approx 1.17\,\mathrm{rad} \qquad \mathbf{2.0}$$

Hence, if the angular distance between holes is β, the asteroid to go through safely, the following condition must be satisfied

$$2\phi = \Delta\gamma + \beta \qquad \mathbf{5.0}$$
$$\therefore \beta = 2\phi - \Delta\gamma$$
$$\beta \approx 0.91\,\mathrm{rad} = 52.3^\circ \qquad \mathbf{1.0}$$

.

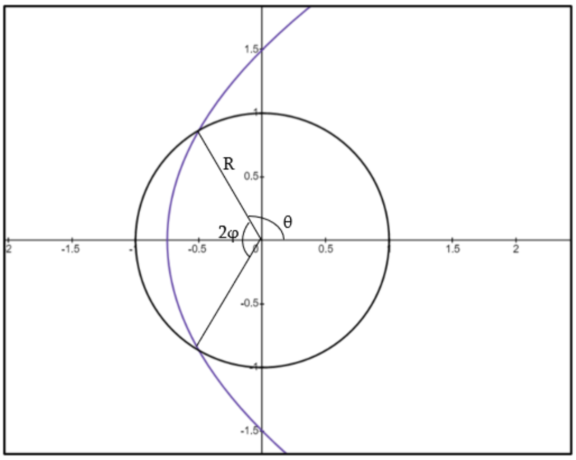


Figure 2: Transit of a asteroid in a polar coordinate system.

(i) By Wien's law, the most probable wavelength radiated by the sphere would be

$$\lambda_1 = \frac{b}{T_1}$$

$$\lambda_2 = \frac{b}{T_2} \qquad \mathbf{2.0}$$

$$\lambda_1' \approx \lambda_1 \left(1 + \frac{dH_0}{c}\right)$$

$$\lambda_2' \approx \lambda_2 \left(1 + \frac{dH_0}{c}\right) \qquad \mathbf{2.0}$$

$$\frac{b}{T_2}\left(1 + \frac{dH_0}{c}\right) < \lambda_{obs} < \frac{b}{T_1}\left(1 + \frac{dH_0}{c}\right) \qquad \mathbf{1.0}$$

12 Solution: Co-orbital satellites (50 Points)

(a) By Kepler's third law we have

$$T_i^2 = \frac{4\pi^2}{GM} r_i^3$$

$$\omega_i = \frac{2\pi}{T_i} = \sqrt{\frac{GM}{r_i^3}} \qquad \mathbf{1.5}$$

$$L_i = m_i \omega_i r_i^2$$

$$\therefore L_i = m_i \sqrt{GMr_i} \qquad \mathbf{1.5}$$

(b) When the satellites are opposing positions (i.e. $\theta = \pi$), we have

$$r_1 = R \pm \frac{x_1}{2}, \;\; r_2 = R \mp \frac{x_2}{2} \qquad \mathbf{2.0}$$

Thus, using the conservation of angular momentum

$$L_{tot} = m_1\sqrt{GM\left(R+\frac{x_1}{2}\right)} + m_2\sqrt{GM\left(R-\frac{x_2}{2}\right)}$$

$$= m_1\sqrt{GM\left(R-\frac{x_1}{2}\right)} + m_2\sqrt{GM\left(R+\frac{x_2}{2}\right)} \qquad \mathbf{4.0}$$

$$\therefore m_1\left(1+\frac{x_1}{4R}\right) + m_2\left(1-\frac{x_2}{4R}\right) = m_1\left(1-\frac{x_1}{4R}\right) + m_2\left(1+\frac{x_2}{4R}\right)$$

$$m_1\frac{x_1}{2R} = m_2\frac{x_2}{2R}$$

$$\therefore \frac{m_1}{m_2} = \frac{x_2}{x_1} \qquad \mathbf{2.0}$$

(c) Let the centre of mass of the ternary system be denoted by C. The total gravitational force on satellite m_2 due to the primary mass M, and the satellite m_1 may be resolved into a radial component, parallel to the line between m_2 and C and a tangential component, perpendicular to the line between m_2 and C. Only the tangential component of the force will act to change the angular momentum of the satellite m_2.
The tangential component of the force is given by

$$F_{2_\perp} = \frac{GMm_2}{r_2^2}\sin(\delta\beta) - \frac{Gm_1m_2}{d^2}\sin\beta \qquad \mathbf{2.0}$$

where $d = dist(m_1, m_2)$, $\beta = \measuredangle m_1m_2C$, and $\delta\beta = \measuredangle Mm_2C$. There are two ways to simplify this expression.
The first way straightforward. By immediately exploiting the fact that m_1, $m_2 \ll M$ and hence $\delta\beta \to 0$, we can say,

$$\frac{d}{\sin\theta} = \frac{r_1}{\sin(\beta+\delta\beta)}$$

$$\sin\beta = \left(\frac{r_1}{d}\right)\sin\theta \qquad \mathbf{3.0}$$

$$\text{Also,}\ \frac{\left(\frac{r_1m_1}{M+m_1}\right)}{\sin(\delta\beta)} = \frac{r_2}{\sin(180-\theta-\delta\beta)}$$

$$\therefore \sin(\delta\beta) = \left(\frac{m_1r_1}{r_2(m_1+M)}\right)\sin\theta \qquad \mathbf{4.0}$$

Another way is using the sine and cosine rules for triangles, we have the expressions

$$\frac{r_2}{\sin(180-\gamma)} = \frac{\left[\left(\frac{r_1m_1}{M+m_1}\right)^2 + r_2^2 - \frac{2m_1r_1r_2}{M+m_1}\cos\theta\right]^{\frac{1}{2}}}{\sin\theta}$$

$$\frac{r_2\sin\theta}{\sin\gamma} = \left[r_2^2 + \left(\frac{r_1m_1}{M+m_1}\right)^2 - \frac{2m_1r_1r_2}{M+m_1}\cos\theta\right]^{\frac{1}{2}}$$

$$\frac{d}{\sin\gamma} = \frac{\frac{Mr_1}{m_1+M}}{\sin\beta}$$

$$\sin\beta = \frac{Mr_1}{m_1+M}\frac{r_2}{d}\sin\theta\left[r_2^2 + \left(\frac{m_1r_1}{m_1+M}\right)^2 - \frac{2m_1r_1r_2}{m_1+M}\cos\theta\right]^{-\frac{1}{2}} \approx \frac{r_1}{d}\sin\theta$$

$$\sin(\delta\beta) = \frac{m_1r_1}{m_1+M}\sin\theta\left[r_2^2 + \left(\frac{m_1r_1}{m_1+M}\right)^2 - \frac{2m_1r_1r_2}{m_1+M}\cos\theta\right]^{-\frac{1}{2}} \approx \frac{m_1r_1}{r_2(m_1+M)}\sin\theta$$

In both cases, finally we obtain

$$F_{2_\perp} = \frac{GMm_2}{r_2^2}\frac{m_1r_1}{r_2(m_1+M)}\sin\theta - \frac{Gm_1m_2}{d^2}\frac{r_1}{d}\sin\theta$$

$$\text{As, } M+m_1 \approx M$$

$$\therefore F_{2_\perp} = Gm_1m_2r_1\sin\theta(r_2^{-3} - d^{-3}) \qquad \mathbf{3.0}$$

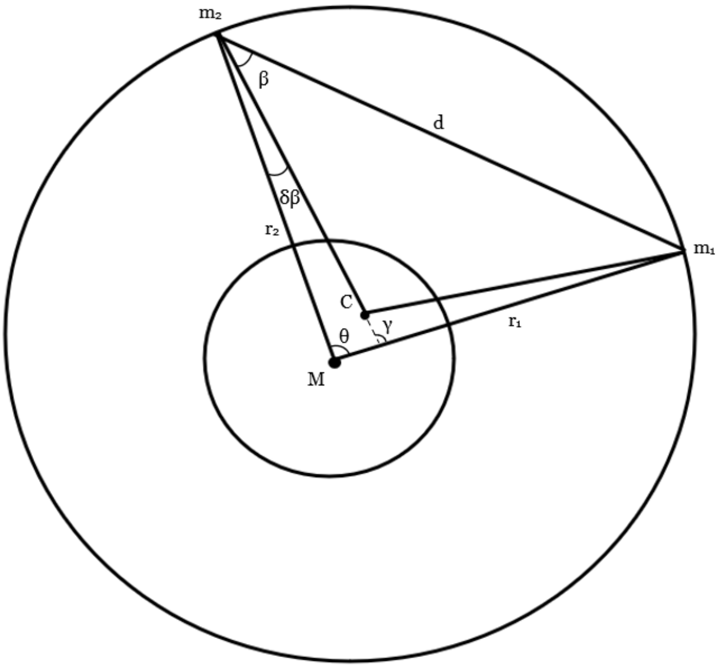


Figure 3: Configuration of the satellites and planet. Circle is centered at C(all bodies are rotating around this point).

The distance between satellites

$$d \approx 2R \sin\left(\frac{\theta}{2}\right) \qquad \mathbf{1.0}$$

The tangential component of the gravitational force then becomes

$$F_{2_\perp} \approx \frac{Gm_1m_2}{R^2} \sin\theta \left(1 - \frac{1}{8\sin^3\left(\frac{\theta}{2}\right)}\right)$$

$$F_{2_\perp} = \frac{Gm_1m_2}{R^2} \left(\sin\theta - \frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)}\right) \qquad \mathbf{2.0}$$

Finally, the torque generated by the gravitational force is then

$$\frac{\Delta L_2}{\Delta t} = F_{2_\perp} r \approx -\frac{Gm_1m_2}{R} \left(\frac{\cos\left(\frac{\theta}{2}\right)}{4\sin^2\left(\frac{\theta}{2}\right)} - \sin\theta\right) \qquad \mathbf{3.0}$$

(d) We can see that

$$\begin{aligned}
\Delta L_i &= m_i\sqrt{GM(r_i+\Delta r_i)} - m_i\sqrt{GMr_i} \\
&= m_i\sqrt{GMr_i}\left[\left(1+\frac{\Delta r_i}{r_i}\right)^{\frac{1}{2}} - 1\right] \\
&= m_i\sqrt{GMr_i}\left(\frac{\Delta r_i}{2r_i}\right) \\
\therefore \frac{\Delta L_i}{\Delta t} &= \frac{1}{2}m_i\sqrt{\frac{GM}{r_i}}\frac{\Delta r_i}{\Delta t} && \mathbf{2.0} \\
\frac{\Delta r_i}{\Delta t} &= \frac{2}{m_i}\sqrt{\frac{r_i}{GM}}\frac{\Delta L_i}{\Delta t} \\
\therefore \frac{\Delta r_2}{\Delta t} &\approx -\frac{2}{m_2}\sqrt{\frac{r_2}{GM}}\frac{Gm_1m_2}{R}h(\theta) \\
\frac{\Delta r_2}{\Delta t} &\approx -2m_1\sqrt{\frac{G}{MR}}h(\theta) && \mathbf{1.0}
\end{aligned}$$

Again, since $r1 \approx r2 \approx R$, we can say by symmetry/the conservation of angular momentum

$$\begin{aligned}
\frac{\Delta r_1}{\Delta t} &\approx 2m_2\sqrt{\frac{G}{MR}}h(\theta) && \mathbf{3.0} \\
\therefore \frac{\Delta s}{\Delta t} &= \frac{\Delta r_2}{\Delta t} - \frac{\Delta r_1}{\Delta t} \\
\frac{\Delta s}{\Delta t} &= -2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta) && \mathbf{2.0}
\end{aligned}$$

(e) Since θ is the angle between the two satellites, we have

$$\begin{aligned}
\frac{\Delta\theta}{\Delta t} &= \omega_2 - \omega_1 = \sqrt{\frac{GM}{r_2^3}} - \sqrt{\frac{GM}{r_1^3}} && \mathbf{2.0} \\
&= \sqrt{\frac{GM}{R^3}}\left[\left(\frac{r_2}{R}\right)^{-\frac{3}{2}} - \left(\frac{r_1}{R}\right)^{-\frac{3}{2}}\right] \\
&= \sqrt{\frac{GM}{R^3}}\left[\left(1+\frac{r_2-R}{R}\right)^{-\frac{3}{2}} - \left(1+\frac{r_1-R}{R}\right)^{-\frac{3}{2}}\right] \\
&= \sqrt{\frac{GM}{R^3}}\left[1 - \frac{3}{2}\frac{(r_2-R)}{R} - 1 + \frac{3}{2}\frac{(r_1-R)}{R}\right] \\
&\approx \sqrt{\frac{GM}{R^3}}\left[\frac{3}{2}\frac{r_1-r_2}{R}\right] \\
\frac{\Delta\theta}{\Delta t} &\approx -\frac{3}{2}\sqrt{\frac{GM}{R^3}}\frac{s}{R} && \mathbf{3.0}
\end{aligned}$$

(f) Using the expressions in the previous two parts,

$$\begin{aligned}
\frac{\Delta s}{\Delta\theta} &= \frac{\Delta s}{\Delta t}\cdot\frac{\Delta t}{\Delta\theta} && \mathbf{1.0} \\
&= 2\sqrt{\frac{G}{MR}}(m_1+m_2)h(\theta)\cdot\frac{2}{3}\sqrt{\frac{R^3}{GM}}\frac{R}{s} \\
&= \frac{4}{3}\frac{R^2}{M}(m_1+m_2)h(\theta)\left(\frac{1}{s}\right) \\
\therefore s\Delta s &= \frac{4R^2}{3}\frac{(m_1+m_2)}{M}h(\theta)\Delta\theta && \mathbf{1.0}
\end{aligned}$$

(g) The minimum distance of 13 000 km corresponds to a minimum angle of

$$\theta_{min} \approx \frac{13000}{150000} \approx 0.0868\,\mathrm{rad} = 4°58' \qquad \mathbf{2.0}$$

Then we substitute the given values the into given expression and the result of (b) which gives

$$\frac{m_2}{m_1} \approx 3.6 \qquad \textbf{1.0}$$

and

$$m_1 + m_2 \approx 2.5 \times 10^{18}\,\mathrm{kg} \qquad \textbf{2.0}$$

Finally

$$m_1 \approx 5.3 \times 10^{17}\,\mathrm{kg} \quad m_2 \approx 1.9 \times 10^{18}\,\mathrm{kg} \qquad \textbf{1.0}$$

13 Solution: Relativistic Beaming (50 Points)

(a) Energy of a photon:

$$E = hf$$

while the momentum:

$$\vec{p} = \frac{hf}{c}\vec{n} \qquad \textbf{2.0}$$

where $\vec{n}$ is an unit vector in the direction of the motion of the photon. Substituting this into energy-momentum transformation law we have:

$$\begin{aligned}
\frac{hf_L}{c} &= \gamma\left(\frac{hf_R}{c} + p_{x_R}\frac{v}{c}\right)\\
p_{x_L} &= \gamma\left(p_{x_R} + \frac{hf_R v}{c^2}\right)\\
p_{y_L} &= p_{y_R}\\
p_{z_L} &= p_{z_R}
\end{aligned} \qquad \textbf{2.0}$$

in our case:

$$\begin{aligned}
p_{x_L} &= |p_L|\cos\theta_L, & p_{y_L} &= |p_L|\sin\theta_L, & p_{z_L} &= 0\\
p_{x_R} &= |p_R|\cos\theta_R, & p_{y_R} &= |p_R|\sin\theta_R, & p_{z_R} &= 0
\end{aligned} \qquad \textbf{3.0}$$

where

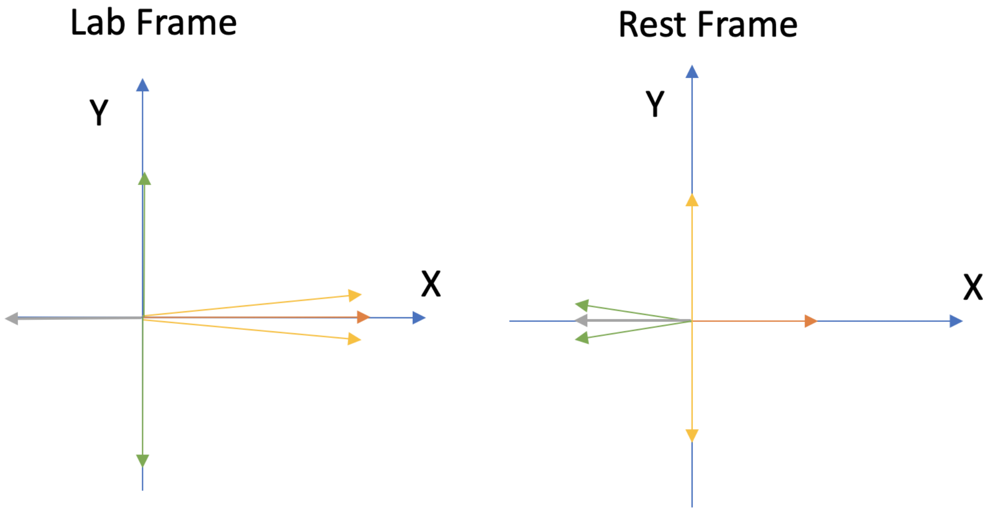
$$|p| = \frac{hf}{c}$$

Thus,

$$\begin{aligned}
f_L &= \gamma\left(f_R + \frac{v}{h}|p_R|\cos\theta_R\right)\\
&= \gamma\left(f_R + \frac{v}{h}\frac{hf_R}{c}\cos\theta_R\right)\\
\therefore f_L &= \gamma f_R\left(1 + \frac{v}{c}\cos\theta_R\right)
\end{aligned} \qquad \textbf{2.0}$$

$$\begin{aligned}
cos(\theta_L) &= \frac{p_{x_L}}{|p_L|} = \frac{cp_{x_L}}{hf_L}\\
&= \frac{c\gamma\left(p_{x_R} + \frac{hf_R v}{c^2}\right)}{h\gamma f_R\left(1 + \frac{v}{c}\cos\theta_R\right)}\\
&= \frac{c\left(\frac{hf_R}{c}\cos\theta_R + \frac{hf_R v}{c^2}\right)}{hf_R\left(1 + \frac{v}{c}\cos\theta_R\right)}\\
\therefore \cos\theta_L &= \frac{\cos\theta_R + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_R}
\end{aligned} \qquad \textbf{2.0}$$

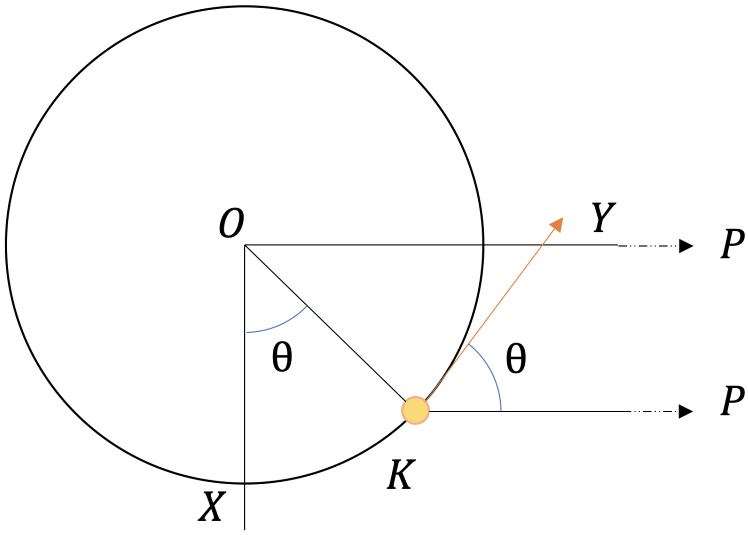
(b) Now just by plugging in the values of cosine we see that in case (i) (orange arrow) $\cos\theta_L = 1$, So the photon keeps moving in the same direction, in case (ii) (Green arrow) $\cos\theta_L = 0$, in case (iii) (Yellow arrow) $\cos\theta_L = v/c$,(iv) (Grey arrow) $\cos\theta_L = -1$.



Marking scheme,
0.5 pt for each answer in the rest frame and lab frame.
(c) From the figure we see that:

$$KP = OP - OK \sin\theta = R - r\sin\theta \qquad \textbf{1.0}$$

where $\theta = \omega t$.



Let $t_L = T$ be the time at which photon reaches P after leaving K at a time t (in lab frame) then:

$$c(T - t) = KP = R - r\sin\theta \approx R$$
$$\therefore t = t_L - \frac{R}{c} \qquad \textbf{1.0}$$

This photon makes angle $\theta = \omega t$ to the direction of its motion in lab frame. Now,

$$\begin{aligned}
\cos\theta_L &= \frac{\cos\theta_R + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_R} \\
\cos\theta_L + \frac{v}{c}\cos\theta_R\cos\theta_L &= \cos\theta_R + \frac{v}{c} \\
\cos\theta_R(1 - \frac{v}{c}\cos\theta_L) &= \cos\theta_L - \frac{v}{c} \\
\therefore \cos\theta_R &= \frac{\cos\theta_L - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta_L}
\end{aligned}$$

2.0

$$\begin{aligned}
f_L &= \gamma f_R\left(1 + \frac{v}{c}\cos\theta_R\right) \\
&= \gamma f_R\left(1 + \frac{\frac{v}{c}\left(\cos\theta_L - \frac{v}{c}\right)}{1 - \frac{v}{c}\cos\theta_L}\right) \\
&= \gamma f_R\left(\frac{1 - \frac{v}{c}\cos\theta_L + \frac{v}{c}\cos\theta_L - \left(\frac{v}{c}\right)^2}{1 - \frac{v}{c}\cos\theta_L}\right) \\
&= \frac{f_R}{\gamma\left(1 - \frac{v}{c}\cos\theta_L\right)} = \frac{f_R}{\gamma\left(1 - \frac{v}{c}\cos(\omega t)\right)} \\
\therefore f_L &= \frac{f_R}{\gamma\left(1 - \frac{v}{c}\cos\left[\omega\left(t_L - \frac{R}{c}\right)\right]\right)}
\end{aligned}$$

3.0

(d)

$$\begin{aligned}
\Delta\Omega_R &= -\Delta(\cos\theta_R)\cdot\Delta\phi \\
&= [\cos\theta_R - \cos(\theta_R + \Delta\theta_R)]\cdot\Delta\phi \\
&= [\cos\theta_R - \cos\theta_R\cos(\Delta\theta_R) + \sin\theta_R\sin(\Delta\theta_R)]\cdot\Delta\phi \\
&= [\cos\theta_R - \cos\theta_R\cdot(1) + \sin\theta_R(\Delta\theta_R)]\cdot\Delta\phi \\
\Delta\Omega_R &= \sin\theta_R(\Delta\theta_R)(\Delta\phi) \\
\therefore \Delta\Omega_L &= \sin\theta_L(\Delta\theta_L)(\Delta\phi)
\end{aligned}$$

3.0

$$\begin{aligned}
\sin^2\theta_L &= 1 - \cos^2\theta_L \\
&= 1 - \left(\frac{\cos\theta_R + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_R}\right)^2 \\
&= \frac{1 + 2\frac{v}{c}\cos\theta_R + \left(\frac{v}{c}\right)^2\cos^2\theta_R - \cos^2\theta_R - 2\frac{v}{c}\cos\theta_R - \left(\frac{v}{c}\right)^2}{\left(1 + \frac{v}{c}\cos\theta_R\right)^2} \\
&= \frac{\left(1 - \left(\frac{v}{c}\right)^2\right)(1 - \cos^2\theta_R)}{\left(1 + \frac{v}{c}\cos\theta_R\right)^2} = \frac{\sin^2\theta_R}{\gamma^2\left(1 + \frac{v}{c}\cos\theta_R\right)^2} \\
\sin\theta_L &= \frac{\sin\theta_R}{\gamma\left(1 + \frac{v}{c}\cos\theta_R\right)}
\end{aligned}$$

3.0

$$\text{Also, } \Delta\theta_L \approx \sin(\Delta\theta_L) = \frac{\sin(\Delta\theta_R)}{\gamma\left(1+\frac{v}{c}\cos\theta_R\right)}$$

$$\Delta\theta_L \approx \frac{\Delta\theta_R}{\gamma\left(1+\frac{v}{c}\cos\theta_R\right)} \qquad \textbf{2.0}$$

$$\frac{\Delta\Omega_L}{\Delta\Omega_R} = \frac{\sin\theta_L(\Delta\theta_L)(\Delta\phi)}{\sin\theta_R(\Delta\theta_R)(\Delta\phi)}$$

$$= \frac{\sin\theta_R(\Delta\theta_R)}{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2}\cdot\frac{1}{\sin\theta_R(\Delta\theta_R)}$$

$$\therefore \Delta\Omega_L = \frac{\Delta\Omega_R}{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2} \qquad \textbf{2.0}$$

(e) Let $N(\theta_L)\Delta\Omega_L\Delta T$ be the number of photons arriving in the vicinity of P within the element of solid angle $\Delta\Omega_L$ and time interval $T, T+\Delta T$ (it is important that $\Delta T \neq \Delta t$ where Δt is the emission time of the same photons (in lab frame)). These being photons emitted by K into the solid angle $\Delta\Omega$ in the time interval $t_{0R}, t_{0R}+\Delta t_{0R}$, so that:

$$N(\theta_L)\Delta\Omega_L\Delta T = N\Delta\Omega_R\Delta t_{0R} \qquad \textbf{2.0}$$

From an earlier part:

$$cT = ct + R - r\sin\theta_L = ct + R - r\sin(\omega t)$$

$$\therefore c\Delta T = c\Delta t - r[\sin(\omega t)\cos(\omega\Delta t) + \sin(\omega\Delta t)\cos(\omega t) - sin(\omega t)]$$

$$c\Delta T = c\Delta t - r\omega\cos(\omega t)\Delta t \qquad \textbf{2.0}$$

$$\frac{\Delta T}{\Delta t_{0R}} = \frac{\Delta t}{\Delta t_{0R}}\frac{\Delta T}{\Delta t} = \gamma\left(1-\frac{v}{c}\cos(\omega t)\right)$$

$$\frac{\Delta t_{0R}}{\Delta T} = \frac{1}{\gamma\left(1-\frac{v}{c}\cos(\omega t)\right)} \qquad \textbf{3.0}$$

$$\frac{N(t_L)}{N_R} = \frac{\Delta\Omega_R\Delta t_{0R}}{\Delta\Omega_L\Delta T}$$

$$= \frac{\gamma^2\left(1+\frac{v}{c}\cos\theta_R\right)^2}{\gamma\left(1-\frac{v}{c}\cos(\omega t)\right)}$$

$$= \frac{\gamma\left(1+\left(\frac{v}{c}\right)\frac{\cos\theta_L-\frac{v}{c}}{1-\frac{v}{c}\cos\theta_L}\right)^2}{\left(1-\frac{v}{c}\cos\theta_L\right)}$$

$$= \frac{\gamma}{\gamma^4\left(1-\frac{v}{c}\cos\theta_L\right)^3}$$

$$N(t_L) = \frac{N_R}{\gamma^3\left(1-\frac{v}{c}\cos\left(t_L-R/c\right)\right)^3} \qquad \textbf{3.0}$$

In terms of energy fluxes

$$F_0 = \frac{hf_R N_R}{R^2} = \frac{L}{4\pi R^2}$$

$$F(t_L) = \frac{hf_L N(t_L)}{R^2} \qquad \textbf{2.0}$$

$$\frac{F(t_L)}{F_0} = \frac{hf_L N(t_L)}{hf_R N_R}$$

$$= \frac{1}{\gamma^4 \left(1 - \frac{v}{c}\cos\left[\omega\left(t - \frac{R}{c}\right)\right]\right)^4}$$

$$F(t_L) = \frac{L}{4\pi R^2 \gamma^4 \left(1 - \frac{v}{c}\cos\left[\omega\left(t - \frac{R}{c}\right)\right]\right)^4} \qquad \textbf{3.0}$$

The radiation is strongly beamed in the direction of motion of the source so that a remote observer in or near the orbital plane of the source sees strongly pulsed radiation.
(f) The amplification (for a given v) is highest when $\cos\theta_L = 1$. **1.0**

$$F(t_L) = \frac{F_0}{\left[\gamma\left(1 - \frac{v}{c}\right)\right]^4}$$

$$A_{\max} = \frac{1}{[0.0975 \times 0.05]^4}$$

$$A_{\max} \approx 1.8 \times 10^9 \qquad \textbf{1.0}$$

$$\text{Similarly, } A_{\min} = \frac{1}{[0.0975 \times 1.95]^4}$$

$$A_{\min} \approx 770 \qquad \textbf{1.0}$$