1. Famous astronomical events

10 p

Correct solution: (10 p)

1. 1764: Discovery of the first planetary nebula
2. 1781: Discovery of Uranus
3. 1801: Discovery of Ceres
4. 1838: First successful measurement of a stellar parallax
5. 1877: Discovery of Phobos and Deimos
6. 1929: Discovery of the expansion of the Universe
7. 1943: Discovery of stellar populations
8. 1963: First identification of a quasar with an optical source
9. 1976: Viking probes arrived at planet Mars
10. 1986: Latest perihelion of comet 1/P Halley
11. 1990: Launch of the Hubble Space Telescope

- 1990: Launch of the Hubble Space Telescope 11
- 1976: Viking probes arrived at planet Mars 9
- 1877: Discovery of Phobos and Deimos 5
- 1986: Latest perihelion of comet 1/P Halley 10
- 1801: Discovery of Ceres (asteroid / dwarf planet) 3
- 1781: Discovery of Uranus (planet) 2
- 1838: First successful measurement of a stellar parallax 4
- 1764: Discovery of the first planetary nebula 1
- 1943: Discovery of stellar populations 7
- 1963: First identification of a quasar with an optical source 8
- 1929: Discovery of the expansion of the Universe 6

The score of the faultless order is 10 points. The list will be checked pairwise. In each case when a later event precedes an earlier event in your solution, one point will be deducted.

2. Deflection of radio photons in the gravitational field of solar system bodies

10 p

One has to notice that in the case of the Sun $d = R_\odot$, that is d equals the solar radius. (1 p)

The angular deflection scales as m/r (mass over radius) of the deflector body, so one has to consider $M_\mathrm{b}/R_\mathrm{b}$, where b stands for Jupiter and Moon, respectively.

(2.1) $$\frac{\Delta\theta_\mathrm{b}}{\Delta\theta_\odot} = \frac{M_\mathrm{b}/R_\mathrm{b}}{M_\odot/R_\odot}$$ (2 p)

This means that:

(2.2) $$\Delta\theta_\mathrm{b} = \Delta\theta_\odot \frac{M_\mathrm{b}/R_\mathrm{b}}{M_\odot/R_\odot}$$ (1 p)

a) Jupiter

Substituting the masses and radii one gets:

$$\Delta\theta_\mathrm{J} = \boxed{0.017''} = \boxed{17\,\mathrm{mas}}$$ (2 p)

The deflection of radio waves by Jupiter's gravitational field is *greater* than 0.1 mas, so it is *observable* with the VLBI network. The answer is: YES (1 p)

b) Moon

Substituting the masses and radii one gets:

$$\Delta\theta_\mathrm{M} = \boxed{0.000\,026''} = \boxed{0.026\,\mathrm{mas}}$$ (2 p)

The deflection of radio waves by Moon's gravitational field is *smaller* than 0.1 mas, so it is *not observable* with the VLBI network. The answer is: NO (1 p)

3. The supermassive black hole in the centre of Milky Way Galaxy and M87 **10 p**

a) The scale of the black hole shadow is set by the Schwarzschild radius $R_S = 2GM/c^2$. The light ring radius is $3R_S$. (2 p)

For a diffraction limited instrument the angular resolution is bounded by

(3.1) $$\theta_{res} = \frac{2R}{d} \geq \frac{1.22\lambda}{D}$$ (2 p)

where λ is the wavelength and D is the diameter of the instrument.

Using the expression for the Schwarzschild radius:

(3.2) $$D \geq \frac{1.22\lambda}{6R_S} d = \boxed{\frac{1.22c^2\lambda d}{12GM}}$$ (2 p)

b) For $\lambda = 1.3\,\mathrm{mm}$, this gives

(a) $D > \boxed{7200\,\mathrm{km}} \approx \boxed{1.1\,\mathrm{R}_\oplus}$ (1 p)

(b) $D > \boxed{6400\,\mathrm{km}} \approx \boxed{1\,\mathrm{R}_\oplus}$ (1 p)

c) (B) Interferometry with array of radio telescopes (2 p)

4. Improving a common reflecting telescope

10 p

a) Let us designate the (unknown) original limiting magnitude of this telescope as m_1, and refer to the intensity (at the focal plane) as I_1. The new limiting magnitude is m_2. Due to the additional intensity, which is due to the better reflectivity of the mirror surfaces, the stellar intensity which will result in the same limiting illumination at the focal plane, will be proportionally smaller, with the ratio of the reflectivities of the new and old mirror coatings:

$$(4.1)\quad I_2 = \frac{\varepsilon_1 \varepsilon_1}{\varepsilon_2 \varepsilon_2} I_1 \qquad (2\,\text{p})$$

The reflectivities should be multiplied twice, since the light coming from the star, loses some energy twice, suffering reflection on each of the two mirrors.

Now applying the well-known magnitude formula:

$$(4.2)\quad m_1 - m_2 = -2.5 \log \frac{I_1}{I_2} = -2.5 \log \frac{\varepsilon_2^2}{\varepsilon_1^2} \qquad (2\,\text{p})$$

With numerical values:

$$m_1 - m_2 = -0.16^{\mathrm{m}}$$

And thus:

$$m_2 = \boxed{m_1 + 0.16^{\mathrm{m}}} \qquad (1\,\text{p})$$

b) **Yes, it is well appreciable by most human eyes.** Therefore, especially the deep-sky-object hunters are using higher reflectivity mirrors in their telescopes. (2 p)

c) The similar way of thinking gives:

$$(4.3)\quad m_2 = m_1 + 2.5 \log \frac{I_1}{I_2} = m_1 + 2.5 \log \frac{\varepsilon_2^2 \varepsilon_3}{\varepsilon_1^3} \qquad (2\,\text{p})$$

With numerical values:

$$m_1 - m_2 = -0.25^{\mathrm{m}}$$

And thus:

$$m_2 = \boxed{m_1 + 0.25^{\mathrm{m}}} \qquad (1\,\text{p})$$

5. Cosmic Microwave Background Oven

10 p

a) If the body is spherical, its radius r can be calculated as:

(5.1) $$V = \frac{m}{\varrho} = \frac{4}{3} r^3 \pi \rightarrow r = \left(\frac{3m}{4\pi\varrho}\right)^{1/3}$$

The area of the body:

(5.2) $$A = 4r^2\pi = 4\pi \left(\frac{3m}{4\pi\varrho}\right)^{2/3} = (4\pi)^{1/3} \left(\frac{3m}{\varrho}\right)^{2/3}$$ (1 p)

According to the Stefan-Boltzmann law, the total energy radiated per unit surface area across all wavelengths per unit time:

(5.3) $$j = \sigma T_{\mathrm{CMB}}^4$$ (1 p)

The CMB is isotropic, which means the absorbed radiation by unit surface area of the spherical body is the same independent of the surface's normal direction. (1 p)

The total absorbed energy by the spherical body per unit time:

(5.4) $$P = jA = \sigma T_{\mathrm{CMB}}^4 \times (4\pi)^{1/3} \left(\frac{3m}{\varrho}\right)^{2/3}$$ (1 p)

Its numerical value:

$$P \approx \boxed{2.3 \times 10^{-6}\,\mathrm{W}}$$ (1 p)

b) The mean energy of CMB photons:

(5.5) $$\langle\varepsilon\rangle = 3kT_{\mathrm{CMB}} = 1.1 \times 10^{-22}\,\mathrm{J}$$ (1 p)

The number of CMB photons the body would absorb per second:

(5.6) $$n = \frac{P}{\langle\varepsilon\rangle}$$ (1 p)

Its numerical value:

$$n = \boxed{2.1 \times 10^{16}\,\mathrm{s}^{-1}}$$ (1 p)

c) Denote the time necessary for raising the temperature by Δt, so:

(5.7) $$P\Delta t = Cm\Delta T \rightarrow \Delta t = \frac{Cm\Delta T}{P}$$ (1 p)

Its numerical value:

$$\Delta t \approx \boxed{1.1 \times 10^{11}\,\mathrm{s}} \approx 1\,253\,200\,\mathrm{d} \approx \boxed{3430\,\mathrm{yr}}$$ (1 p)

6. The height of the chimney of Tiszaújváros power plant

20 p

a) (B) the one on December 16 (1 p)

At a given moment of the day, the Sun elevation is lower in winter on the northern hemisphere.

b) (B) late in the morning (1 p)

The shadows point slightly west of north. In fact these satellite images are taken at around 9:45 UTC (10:45 Central European Time).

c) The measured shadow lengths are $x_1 = 125\,\mathrm{m}$ (epoch 1, summer solstice) and $x_2 = 780\,\mathrm{m}$ (epoch 2, winter solstice).

Let the unknown height of the chimney be h. The chimney and its shadow form the legs of a right-angled triangle and thus

$$\tan\alpha_1 = h/x_1 \tag{6.1}$$

$$\tan\alpha_2 = h/x_2, \tag{6.2}$$ (1 p)

where α_1 and α_2 are the Sun elevation angles at epoch 1 and epoch 2, respectively. (1 p)

At the time of the summer and winter solstices, the Earth is located at opposite sides of the Sun during its orbital revolution. Since the Equator is tilted by $\varepsilon \approx 23.5°$ to the orbital plane, at a given point of the surface, and at a given time of the day, the Sun elevation angles will differ by $\alpha_1 - \alpha_2 = 2\varepsilon$ (here we ignore refraction and that the satellite images were not taken on the exact solstice days). (4 p)

Notice that, to a good approximation, $\alpha_1 - \alpha_2 \approx 45°$. Assuming $\alpha_1 = \alpha_2 + 45°$ makes the solution a lot easier. Using the trigonometric identity

$$\tan\alpha_1 = \tan(\alpha_2 + 45°) = \frac{\tan\alpha_2 + \tan 45°}{1 - \tan\alpha_2 \tan 45°}, \tag{6.3}$$

and substituting $\tan 45° = 1$, Eq. (6.3) becomes

$$\tan\alpha_1 = \frac{1 + \tan\alpha_2}{1 - \tan\alpha_2}. \tag{6.4}$$ (4 p)

Substituting $\tan\alpha_1$ and $\tan\alpha_2$ from Eqs. (6.1) and (6.2) into Eq. (6.4), one gets a quadratic equation for the unknown h:

$$\frac{h}{x_1} = \frac{1 + \frac{h}{x_2}}{1 - \frac{h}{x_2}}. \tag{6.5}$$ (2 p)

Reducing this equation to the standard format

$$h^2 + h(x_1 - x_2) + x_1 x_2 = 0, \tag{6.6}$$

and according to the quadratic formula, the two roots are obtained as

$$h_{1,2} = \frac{x_2 - x_1 \pm \sqrt{(x_1 - x_2)^2 - 4x_1x_2}}{2}. \tag{6.7}$$ (1 p)

Now with the measured values of $x_1 = 125\,\mathrm{m}$ and $x_2 = 780\,\mathrm{m}$, the two roots for h are $h_1 = 426\,\mathrm{m}$ and $h_2 = 229\,\mathrm{m}$. (1 p)

Obviously, only one of the solutions is valid for our case.

One may simply notice that $h_1 \approx 426\,\mathrm{m}$ seems just too high for a chimney (would in fact be comparable to the Empire State Building in New York, with 443 m; the Eiffel tower in Paris is only 324 m high).

Alternatively, substituting $h_1 = 426\,\mathrm{m}$ into Eq. (6.2), we obtain the Sun elevation angle at around the winter solstice, $\alpha_2 = \arctan(h_1/x_2) \approx 28.7°$. However, at the geographic latitudes of Hungary, the Sun does not rise above $\approx 19°$ at this time of the year, so the larger value obtained for h is clearly invalid. (2 p)

Note that according to the official data, the Tiszaújváros power plant has a chimney as high as $h = 250\,\mathrm{m}$.

d) (B) and (E) (2 p)

7. Effect of sunspots on solar irradiation

20 p

a) The Sun's total power can be calculated from the Stefan-Boltzmann law:

(7.1) $P = A\sigma T^4$, (1 p)

where A is the emitting surface, σ is the Stefan-Boltzmann constant and T is the (effective) temperature.

The total power emitted homogeneously and isotropically by the Sun ($P_\odot$) distributes uniformly on the surface area of a sphere with radius $d_\oplus = 1\,\mathrm{au}$. Hence:

(7.2) $$I_\odot = \frac{P_\odot}{4\pi d_\oplus^2} = \frac{\sigma 4\pi R_\odot^2 T_\odot^4}{4\pi d_\oplus^2} = \sigma \frac{R_\odot^2 T_\odot^4}{d_\oplus^2},$$ (2 p)

where $R_\odot$ and $T_\odot$ denote the radius and effective temperature of the Sun, respectively.

With the numerical values taken from the *Table of constants*:

$$I_\odot = \boxed{1365.9\,\mathrm{W\,m^{-2}}}$$ (1 p)

b) In the first days of January the Earth is in perihelion, while in the beginning of July it is in aphelion. Therefore, instead of the mean Earth-Sun distance (i.e. the semi-major axis of the Earth's orbit), now one should substitute the perihelion and aphelion distances, respectively. Therefore, one has to write that:

(7.3) $$I_{\odot,\mathrm{perihelion}} = \frac{P_\odot}{4\pi(d_\oplus(1-e))^2} = \sigma\frac{R_\odot^2 T_\odot^4}{(d_\oplus(1-e))^2} = \frac{I_\odot}{(1-e)^2},$$

(7.4) $$I_{\odot,\mathrm{aphelion}} = \frac{P_\odot}{4\pi(d_\oplus(1+e))^2} = \sigma\frac{R_\odot^2 T_\odot^4}{(d_\oplus(1+e))^2} = \frac{I_\odot}{(1+e)^2},$$ (1 p)

where $e = 0.016\,710\,22$ is the eccentricity of the Earth's orbit (see the *Table of constants*).

With numerical values:

in early January: $I_{\odot,\mathrm{perihelion}} = \boxed{1412.7\,\mathrm{W\,m^{-2}}}$ (1 p)

in early July: $I_{\odot,\mathrm{aphelion}} = \boxed{1321.4\,\mathrm{W\,m^{-2}}}$ (1 p)

The ratio of the values: $I_{\odot,\mathrm{perihelion}}/I_{\odot,\mathrm{aphelion}} = \boxed{1.0691}$. The perihelion irradiation is higher by about 7 %. (1 p)

c) In this case the total power of the Sun becomes

(7.5) $$P'_\odot = \sigma\left[T_\odot^4(A_\odot - A_{\mathrm{sp}}) + T_{\mathrm{sp}}^4 A_{\mathrm{sp}}\right],$$ (2 p)

where $A_\odot$ stands for the solar surface, while A_{sp} is the area of the spot (as a circle).

Thus, the solar constant takes the following form:

(7.6) $$I'_\odot = \frac{P'_\odot}{4\pi d_\oplus^2} = \frac{\sigma\pi\left[T_\odot^4(4R_\odot^2 - R_{\mathrm{sp}}^2) + T_{\mathrm{sp}}^4 R_{\mathrm{sp}}^2\right]}{4\pi d_\oplus^2} = \frac{\sigma}{4d_\oplus^2}\left[T_\odot^4(4R_\odot^2 - R_{\mathrm{sp}}^2) + T_{\mathrm{sp}}^4 R_{\mathrm{sp}}^2\right] =$$ (2 p)

$$I_\odot\left[1 - \left(\frac{R_{\mathrm{sp}}}{2R_\odot}\right)^2 + \left(\frac{R_{\mathrm{sp}}}{2R_\odot}\right)^2\left(\frac{T_{\mathrm{sp}}}{T_\odot}\right)^4\right] = I_\odot\left[1 - \left(\frac{R_{\mathrm{sp}}}{2R_\odot}\right)^2\left(1 - \left(\frac{T_{\mathrm{sp}}}{T_\odot}\right)^4\right)\right] = \boxed{0.9991 I_\odot}$$ (2 p)

With numerical value:

$$I'_\odot = \boxed{1364.6\,\mathrm{W\,m^{-2}}}$$ (1 p)

d) As we have seen, the ratio is

$$A = \frac{I'_\odot}{I_\odot} = \frac{1364.64\,\mathrm{W\,m^{-2}}}{1365.92\,\mathrm{W\,m^{-2}}} = \boxed{0.9991},$$

or, converting it into percentage, the solar constant in the case of the spotted Sun will be lower by 0.09 %.

When this sunspot emerges in the surface of the Sun, solar emission no longer will be isotropic, i.e. uniform in every directions. When the sunspot is fully seen from the Earth, the ratio of the irradiation of the Earth in the spotted and unspotted cases will be equal to the ratio of the solar powers in the two cases, i.e.,

(7.7) $P_{\odot,\mathrm{disk}} = \sigma T_\odot^4 A_\odot = \sigma\pi T_\odot^4 R_\odot^2$ (1 p)

(7.8) $P'_{\odot,\mathrm{disk}} = \sigma\left[T_\odot^4 (A_\odot - A_{\mathrm{sp}}) + T_{\mathrm{sp}}^4 A_{\mathrm{sp}}\right] = \sigma\pi\left[T_\odot^4 (R_\odot^2 - R_{\mathrm{sp}}^2) + T_{\mathrm{sp}}^4 R_{\mathrm{sp}}^2\right]$ (1 p)

and therefore, the ratio of the irradiations (or the effective solar constants) can be written as:

(7.9) $A' = \dfrac{P'_{\odot,\mathrm{disk}}}{P_{\odot,\mathrm{disk}}} = \dfrac{T_\odot^4 (R_\odot^2 - R_{\mathrm{sp}}^2) + T_{\mathrm{sp}}^4 R_{\mathrm{sp}}^2}{T_\odot^4 R_\odot^2} = 1 - \left(\dfrac{R_{\mathrm{sp}}}{R_\odot}\right)^2 \left[1 - \left(\dfrac{T_{\mathrm{sp}}}{T_\odot}\right)^4\right]$ (2 p)

With numerical values:

$A' = \boxed{0.9963}$ (1 p)

Thus, when the spot directed towards the Earth the total irradiation of the Earth becomes weaker by 0.37 %.

8. Amplitude variation of RR Lyrae stars

20 p

a) The total power of the star in the given lambda:

(8.1) $$I(\lambda, T, R) = C_1 R^2 F(\lambda, T) = C_1 R^2 \frac{1}{\lambda^5} \exp\left(-\frac{C_b}{\lambda T}\right)$$ (1 p)

Converting to magnitudes:

(8.2) $$m(\lambda, T, R) = -2.5 \log I(\lambda, T) = C_2 + 12.5 \log \lambda - 5 \log R - 2.5 \log e \frac{C_b}{\lambda T}$$ (1 p)

Then:

(8.3) $$A(\lambda) = |m(\lambda, T_{\max}) - m(\lambda, T_{\min})| = \frac{2.5 \log e \, C_b}{\lambda} \left(\frac{1}{T_{\min}} - \frac{1}{T_{\max}}\right)$$ (1 p)

Thus:

(8.4) $$\frac{A(\lambda_1)}{A(\lambda_2)} = \frac{\lambda_2}{\lambda_1}$$ (1 p)

With numerical values:

$$\frac{A(\lambda_1)}{A(\lambda_2)} = \frac{2 \times 10^{-6}\,\mathrm{m}}{5 \times 10^{-7}\,\mathrm{m}} = \boxed{4}$$ (1 p)

b) From Eq. (8.3):

(8.5) $$A_{\max}(\lambda_1) = \frac{2.5 \log e \, C_b}{\lambda_1} \left(\frac{1}{T_{\min}} - \frac{1}{T_{\max}}\right)$$ (2 p)

With numerical values:

$$A_{\max}(\lambda_1) = \frac{2.5 \log e \times 0.0144\,\mathrm{m\,K}}{5 \times 10^{-7}\,\mathrm{m}} \left(\frac{1}{6000\,\mathrm{K}} - \frac{1}{7400\,\mathrm{K}}\right) = \boxed{1^{\mathrm{m}}}$$ (1 p)

c) If we ignore the temperature variation, then:

(8.6) $$A(\lambda) = |m(\lambda, R_{\max}) - m(\lambda, R_{\min})| = 5 \log \frac{R_{\max}}{R_{\min}}$$ (2 p)

With numerical values:

$$A(\lambda) = 5 \log \frac{1.05}{0.9} = \boxed{0.33^{\mathrm{m}}}$$ (1 p)

d) From Eq. (8.4):

(8.7) $$A_{\max}(\lambda_2) = A_{\max}(\lambda_1) \frac{\lambda_1}{\lambda_2} = 1^{\mathrm{m}} \times \frac{1}{4} = 0.25^{\mathrm{m}}$$ (1 p)

Similarly:

$$A_{\min}(\lambda_1) = \frac{2.5 \log e \times 0.0144\,\mathrm{m\,K}}{5 \times 10^{-7}\,\mathrm{m}} \left(\frac{1}{6100\,\mathrm{K}} - \frac{1}{6900\,\mathrm{K}}\right) = 0.6^{\mathrm{m}}$$ (2 p)

and

$$A_{\min}(\lambda_2) = A_{\min}(\lambda_1) \frac{\lambda_1}{\lambda_2} = 0.6^{\mathrm{m}} \times \frac{1}{4} = \boxed{0.15^{\mathrm{m}}}$$ (2 p)

At $\lambda_1 = 500\,\mathrm{nm}$ the amplitude changes by $1^{\mathrm{m}} - 0.6^{\mathrm{m}} = 0.4^{\mathrm{m}}$, while at $\lambda_2 = 2000\,\mathrm{nm}$ it is reduced to $0.25^{\mathrm{m}} - 0.15^{\mathrm{m}} = 0.1^{\mathrm{m}}$, that is a significant difference.

e) B and C (4 p)

9. Distance of the Lagrangian point L_2 of the Earth–Moon system

20 p

The relay satellite revolves on a circular orbit, due to the gravitational forces of the Earth and the Moon. Its net acceleration is:

$$\text{(9.1)}\quad \left(r + \frac{M}{m+M} R\right)\omega^2 = \frac{GM}{(R+r)^2} + \frac{Gm}{r^2}, \qquad \text{(4 p)}$$

where R and r denote the (constant) distances between Earth and Moon, and between Moon and the satellite, respectively, while M and m stand for the masses of Earth and Moon, respectively. Using Kepler's third law, the angular velocity of the satellite, which is equal to the angular velocity of the Moon can be written as

$$\text{(9.2)}\quad \omega^2 = \frac{4\pi^2}{P^2} = \frac{G(M+m)}{R^3}. \qquad \text{(2 p)}$$

Substituting Eq. (9.2) into Eq. (9.1), eliminating the gravitational constant (G), and introducing the dimensionless quantity $x = r/R$, we obtain

$$\text{(9.3)}\quad \frac{(M+m)x + M}{R^2} = \frac{M}{R^2(1+x)^2} + \frac{m}{R^2x^2} \qquad \text{(2 p)}$$

$$\text{(9.4)}\quad (M+m)x + M = \frac{M}{(1+x)^2} + \frac{m}{x^2}. \qquad \text{(2 p)}$$

Assuming that $x = r/R \ll 1$ we can apply the approximation $(1+x)^{-2} \approx 1 - 2x$, which leads to the following expression:

$$\text{(9.5)}\quad (M+m)x + M = M(1-2x) + \frac{m}{x^2}, \qquad \text{(2 p)}$$

from which we get

$$\text{(9.6)}\quad x^3 = \frac{m}{3M+m}. \qquad \text{(2 p)}$$

Substituting the mass ratio $m/M = 0.0123$ of the Earth-Moon system one can obtain

$$\text{(9.7)}\quad x = 0.159\,83, \qquad \text{(2 p)}$$

and therefore

$$\text{(9.8)}\quad r = xR_{\text{Moon}} = 0.159\,83 \times 384\,400\,\text{km} = 61\,440\,\text{km} \qquad \text{(2 p)}$$

From this result we should subtract the radius of the Moon, and such a way we obtain that

$$\text{(9.9)}\quad h = r - R_{\text{Moon}} = 61\,440\,\text{km} - 1737\,\text{km} = \boxed{59\,703\,\text{km}} \qquad \text{(2 p)}$$

Here is another solution, which is erroneous in principle (or say, "approximative") but numerically gives an almost equivalent result.

One might say, that $m \ll M$ and therefore m can be neglected either in the left hand side of Eq. (9.1) and in the r.h.s. of Eq. (9.2), i.e. one can use the following approximation:

$$\text{(9.10)}\quad M + m \approx M$$

In this case instead of Eq. (9.6) one can arrive at

(9.11) $x^3 = \frac{m}{3M}$,

which leads to

(9.12) $x = 0.160\,05$,

and therefore,

$$r = xR_{\text{Moon}} = 0.160\,05 \times 384\,400\,\text{km} = 61\,524\,\text{km}$$

and, finally

$$h = r - R_{\text{Moon}} = 61\,524\,\text{km} - 1737\,\text{km} = \boxed{59\,787\,\text{km}}$$

This latter should be accepted as a full point solution, too. If, however, m is dropped out only from one of the two equations, then at least 2 points should be subtracted.

10. South → East → North

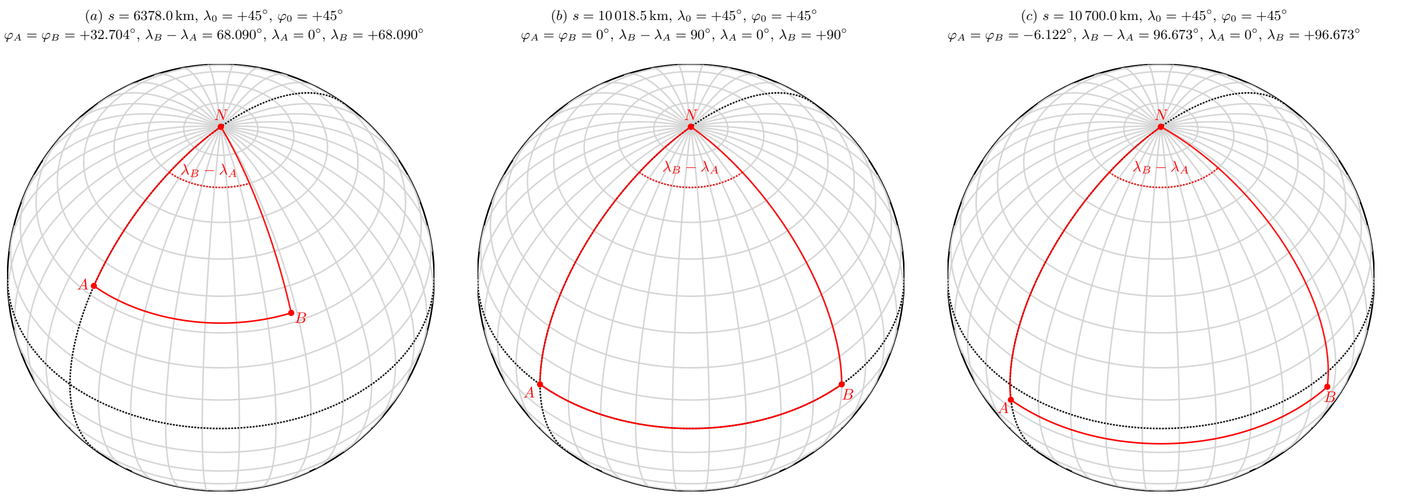
20 p

1. The trivial solution of the problem is the North Pole (N). Starting from here we travel 6378 km to the South ($\widehat{NA}$) along a meridian (e.g. Greenwich meridian), then 6378 km to the East along a parallel ($\widehat{AB}$), and finally 6378 km to the North ($\widehat{BN}$) (see figure (a), where the black dashed lines are the Equator and the Greenwich meridian).

For the general solution, mark the path to be taken in one direction with s and determine what it may be!

Let us denote the longitude and latitude of the point A by λ_A and φ_A, and the point B by λ_B and φ_B.

The triangle ANB is a spherical one only if the points A and B are on the Equator (see figure (b)), i.e. $\varphi_A = \varphi_B = 0°$ and $s = R\pi/2$, because the arc $\widehat{AB}$ is lying along a great circle only in this case, otherwise it is a part of a parallel. The point A and B can be below the Equator as well (figure (c)).



Next, we measure the angles in radians.

The central angle of the arc $\widehat{NA}$ on the meridian fitting to the point A (O is the centre of the sphere):

(10.1) $NOA\sphericalangle = \dfrac{\pi}{2} - \varphi$, where $\varphi = \varphi_A = \varphi_B$

Thus the length s of the arc $\widehat{NA}$:

(10.2) $s = R\left(\dfrac{\pi}{2} - \varphi\right)$, where $s < R\pi$

The radius of the latitude circle fitting to the points A and B:

(10.3) $r_{AB} = R\sin\left(\dfrac{\pi}{2} - \varphi\right) = R\cos\varphi$

The central angle of the arc $\widehat{AB}$ is $\lambda_B - \lambda_A$, thus the length of the arc $\widehat{AB}$:

(10.4) $s = r_{AB}\,(\lambda_B - \lambda_A) = R\cos\varphi\,(\lambda_B - \lambda_A)$

From the Eq. (10.2) and Eq. (10.4):

(10.5) $\varphi = \dfrac{\pi}{2} - \dfrac{s}{R}$ and $\lambda_B - \lambda_A = \dfrac{s}{R\cos\varphi}$

Substitute the expression of φ from the first equation into the second equation:

$$(10.6)\quad \lambda_B - \lambda_A = \frac{s}{R\cos\varphi} = \frac{\frac{s}{R}}{\cos\left(\frac{\pi}{2} - \frac{s}{R}\right)} = \frac{\frac{s}{R}}{\sin\frac{s}{R}}$$

For a given arc length of s ($s < R\pi$) the latitude φ of the points A and B, and the difference between their longitudes we can calculate from the following equations:

$$(10.7)\quad \varphi = \frac{\pi}{2} - \frac{s}{R} \text{ and } \lambda_B - \lambda_A = \frac{\frac{s}{R}}{\sin\frac{s}{R}}$$

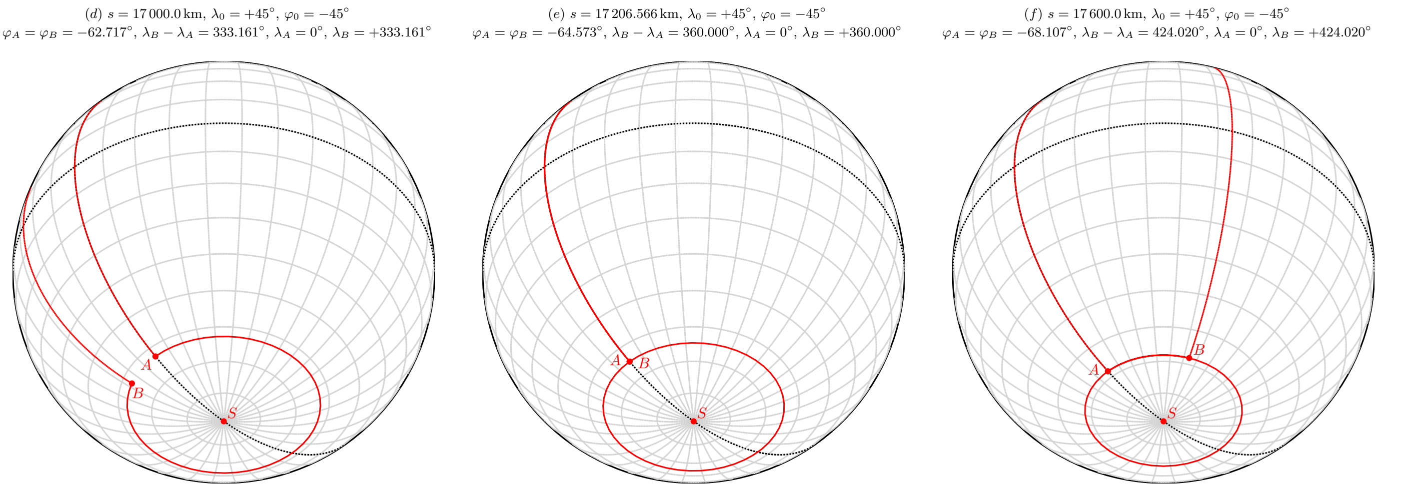
***** Supplement. It is just for the sake of completeness, not necessary for the full marks.

It is well known, that

$$(10.8)\quad \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x}{\sin x} = 1,$$

thus $\lambda_B - \lambda_A = 1$ in case of $s \to 0$.

Therefore the difference in longitude has a lower limit (1), but it has no upper limit. This means, that the distance between the points A and B will increase by increasing s, the difference between their longitudes will approximate 2π (see figure (d)), and in case of $\lambda_B - \lambda_A = 2\pi$ the points A and B will be the same (see figure (e)). With further increasing of s the difference $\lambda_B - \lambda_A$ will be larger than 2π (see figure (f)), thus we have to stop at that point on the circle where the length of the path along it reaches the value of s, and then turn to the North.



The arc length s_{lim} corresponding to the "limit case" of $\lambda_B - \lambda_A = 2\pi$ we can derive from the numerical solution of the following transcendent equation:

$$\frac{s_{\text{lim}}}{R} - 2\pi\sin\frac{s_{\text{lim}}}{R} = 0$$

Thus find the root of the function

$$f(s) = \frac{s}{R} - 2\pi\sin\frac{s}{R},$$

i.e. the value $s_{\rm lim}$, where $f(s_{\rm lim}) = 0$. We can solve it by iteration using e.g. the Newton–Raphson method:

$$s_{k+1} = s_k - \frac{f(s_k)}{f'(s_k)}, \quad k = 0, 1, 2, \ldots$$

The first derivative of the function $f(s)$:

$$f'(s) = \frac{1}{R} - \frac{2\pi}{R} \cos \frac{s}{R}$$

We know that the parallel in question runs south from the Equator, thus a good initial value for the iteration could be $s_0 = R\pi/2 + \Delta s$, where $\Delta s = 1000\,\mathrm{km}$.

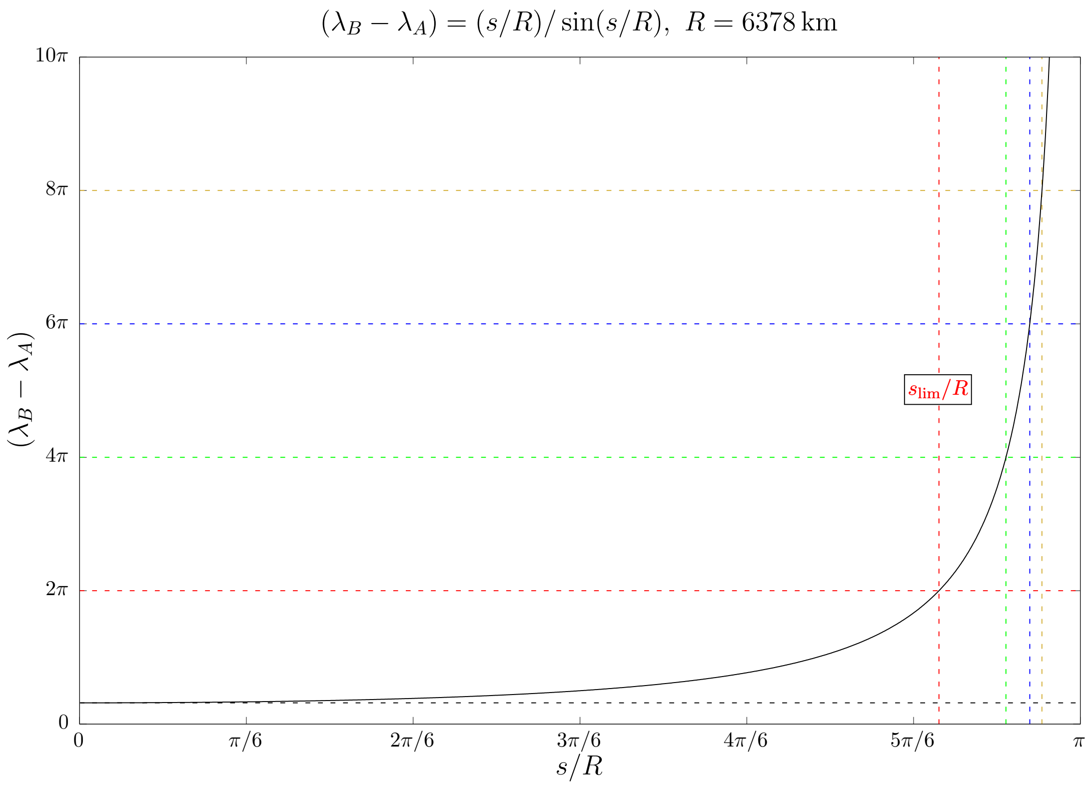
The result of the iteration and the corresponding latitude with three decimal places:

$$s_{\rm lim} = 17\,206.566\,\mathrm{km}, \ \varphi_{\rm lim} = -64.573°$$

$\lambda_B - \lambda_A$ is strictly increasing function of s and

(10.9) $\lim\limits_{s/R \to 0} \lambda_B - \lambda_A = 1, \quad \lim\limits_{s/R \to \pi} \lambda_B - \lambda_A = \infty,$

thus for any values of $0 < s < R\pi$ can be found an appropriate angle $\lambda_B - \lambda_A$ (see the figure below).



***** End of supplement.

In conclusion, the solution in this case is the North Pole (N). (2 p)

For the value of $s = 6378\,\mathrm{km}$ the solution is an (a)-type curve. The coordinates of the turning points are:

$$\lambda_A = \boxed{0°}, \ \lambda_B = \boxed{68.09°}, \ \varphi_A = \varphi_B = \boxed{32.704°}$$ (3 p)

2. The non-trivial solutions are similar to the (e)-type ("lasso") solution of the previous part, only the starting point is not the North Pole, but its location depends on the given value of arc length s. The circle of the lasso will run south from the (e)-type "limit circle", because the circumference of parallels running north from it are greater than the north-south arc.

The procedure is the following.

Knowing s, we have to find the parallel around the South Pole (S), whose circumference is an integer multiple of s. (2 p)

If the corresponding latitude is φ_D, then:

(10.10) $2k\pi R\cos\varphi_D = s,\ k = 1, 2, 3, \ldots$ (5 p)

If the latitude of the sought starting point C is φ_C, then the length s of the arc to be taken along the meridian:

(10.11) $s = R\left(\varphi_C - \varphi_D\right)$

From the Eqs. (10.10) and (10.11):

(10.12) $\varphi_C = \varphi_D + 2k\pi\cos\varphi_D,\ k = 1, 2, 3, \ldots$

Thus:

(10.13) $\varphi_D = -\arccos\dfrac{s}{2k\pi R},\ k = 1, 2, 3, \ldots$ (2 p)

and

(10.14) $\varphi_C = -\arccos\dfrac{s}{2k\pi R} + \dfrac{s}{R},\ k = 1, 2, 3, \ldots$ (2 p)

The negative sign denotes that the parallel is in the southern hemisphere.

So if the starting point is not the North Pole (N), there are infinitely many solutions.

For the value of $s = 6378\,\text{km}$ the first 5 ($k = 1, \ldots, 5$) value for φ_D in degrees with 3 decimal places:

$$\varphi_{D_1} = -80.842°,\ \varphi_{D_2} = -85.436°,\ \varphi_{D_3} = -86.959°,\ \varphi_{D_4} = -87.72°,\ \varphi_{D_5} = -88.176° \quad (2\,\text{p})$$

The first 5 ($k = 1, \ldots, 5$) value for φ_C in degrees with 3 decimal places:

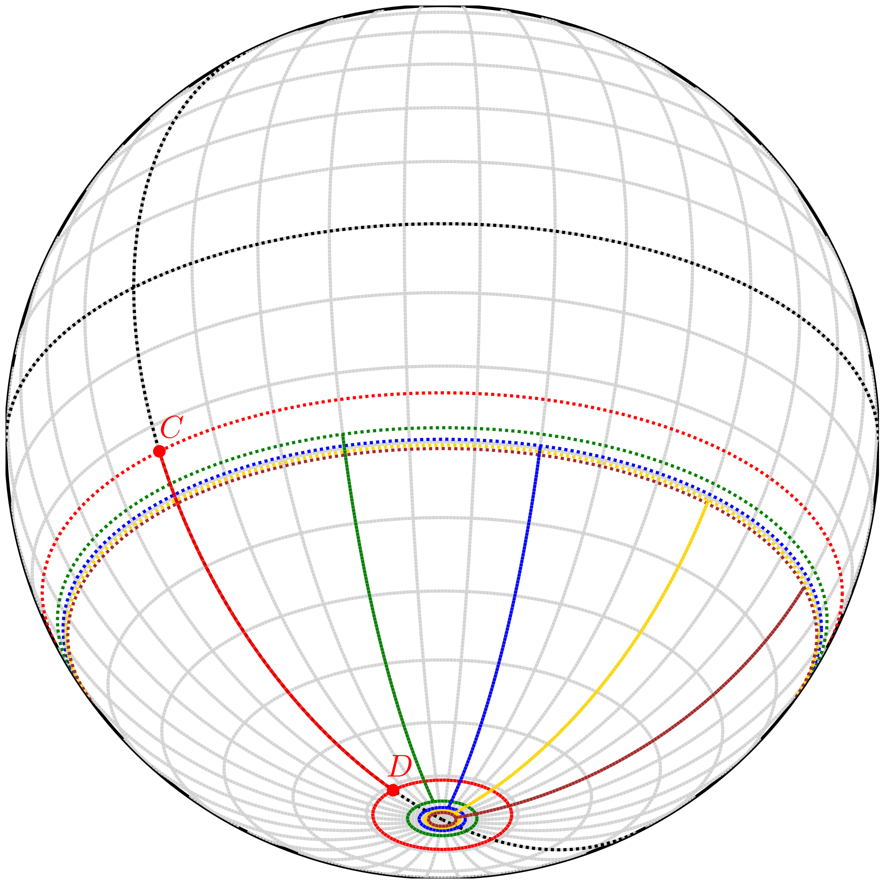
$$\varphi_{C_1} = -23.546°,\ \varphi_{C_2} = -28.14°,\ \varphi_{C_3} = -29.663°,\ \varphi_{C_4} = -30.424°,\ \varphi_{C_5} = -30.88° \quad (2\,\text{p})$$

In conclusion, starting to the South from a point of the parallel marked with a dashed red line (C), we get to a point on a parallel marked with a continuous red line (D). On this parallel we travel a full circle to the East, then we get back to the point D, then turn to North and travel back to starting point C.

For the further values of k the procedure is very similar, but we have to perform two full circles along the continuous green parallel, three along the blue, four along the gold, five along the brown and so on.

Such kind of solution exists for any values of $0 < s < s_{\text{lim}}$.

(g) $s = 6378.0\,\mathrm{km}$, $\lambda_0 = +45°$, $\varphi_0 = -30°$
$\varphi_{D_1} = -80.842°$, $\varphi_{D_2} = -85.436°$, $\varphi_{D_3} = -86.959°$, $\varphi_{D_4} = -87.720°$, $\varphi_{D_5} = -88.176°$
$\varphi_{C_1} = -23.546°$, $\varphi_{C_2} = -28.140°$, $\varphi_{C_3} = -29.663°$, $\varphi_{C_4} = -30.424°$, $\varphi_{C_5} = -30.880°$



11. Identification of light curve type of selected variable stars

25 p

a) The score for each correct answer: 1 point (8 p)

- Heart-beat star — 8
- RR Lyrae type (RRab subclass) pulsating variable star — 4
- Eclipsing binary of Algol type (semi-detached) with a pulsating component — 6
- α^2 CVn star — 5
- W Vir type (Population II) Cepheid pulsating variable — 3
- Detached eclipsing binary with strong reflection effect — 2
- Contact eclipsing binary of W UMa type — 1
- Rotationally variable (spotted) star — 7

b) The values below have been determined by an AoV period searching algorithm and rounded to 2 decimal places. The range of ±5 % is also given for each value: the period in this range is acceptable. The score for each correct answer: 2 points. (16 p)

1.	RW Dor	$P \approx 0.29$ d	$0.28\,\mathrm{d} \leq P \leq 0.30\,\mathrm{d}$
2.	FO Eri	$P \approx 2.20$ d	$2.09\,\mathrm{d} \leq P \leq 2.31\,\mathrm{d}$
3.	UY Eri	$P \approx 2.21$ d	$2.10\,\mathrm{d} \leq P \leq 2.32\,\mathrm{d}$
4.	ST Pic	$P \approx 0.49$ d	$0.47\,\mathrm{d} \leq P \leq 0.51\,\mathrm{d}$
5.	AH Col	$P \approx 1.10$ d	$1.04\,\mathrm{d} \leq P \leq 1.16\,\mathrm{d}$
6.	VV Ori	$P \approx 1.49$ d	$1.42\,\mathrm{d} \leq P \leq 1.56\,\mathrm{d}$
7.	TIC 147272181	$P \approx 0.55$ d	$0.52\,\mathrm{d} \leq P \leq 0.58\,\mathrm{d}$
8.	24 Eri	$P \approx 8.12$ d	$7.71\,\mathrm{d} \leq P \leq 8.53\,\mathrm{d}$

c) (D) Supernova in a distant galaxy (ASASSN-18tb) (1 p)

12. Distance to a Near-Earth Object

25 p

a) The geometric situation is the following: (5 p)

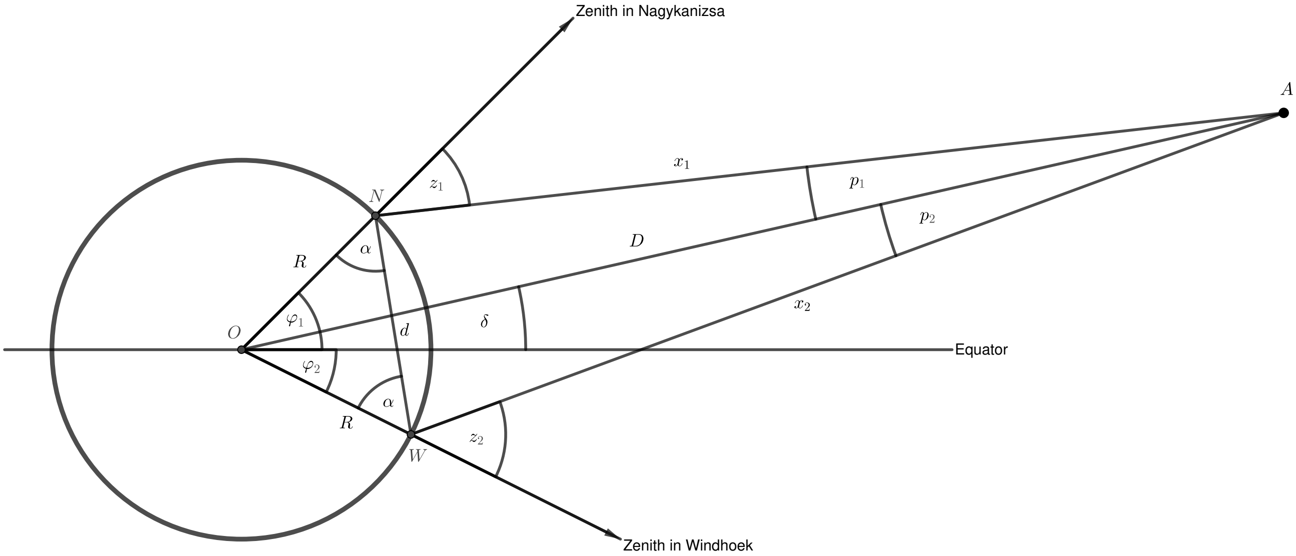


Fig. 12.1: N: Nagykanizsa, W: Windhoek, O: centre of Earth, A: asteroid, D: distance from the centre of Earth to the asteroid $\left(D = \overline{OA}\right)$, R: radius of the Earth

b) One can recognize that the triangle NOW is an isosceles triangle, two sides (ON and OW) are equal to each other in length:

(12.1) $ON = OW = R$ (1 p)

Therefore the angles $\alpha = ONW\sphericalangle = OWN\sphericalangle$. For the NOW triangle we can write, that:

(12.2) $2\alpha + \varphi_1 + \varphi_2 = 180^\circ \rightarrow \alpha = 90^\circ - \dfrac{\varphi_1 + \varphi_2}{2}$ (1 p)

Let us denote the direct – i.e. measured through the body of the Earth, not on the surface – distance between Nagykanizsa and Windhoek by d: $d = NW$. This distance can be obtained straightforward by the cosine theorem (one has to take care that the absolute values of the latitudes should be added inside the cosine function):

(12.3) $d = \sqrt{R^2 + R^2 - 2R^2\cos(\varphi_1 + \varphi_2)} = R\sqrt{2(1 - \cos(\varphi_1 + \varphi_2))}$ (2 p)

With numerical values:

$$d = 1.133\,05\,R$$ (1 p)

Let us use the following notations:

(12.4) $x_1 = \overline{NA},\ x_2 = \overline{WA},\ p = p_1 + p_2$

For the $NOWA$ quadrilateral we can write that:

(12.5) $\varphi_1 + \varphi_2 + (180^\circ - z_1) + p + (180^\circ - z_2) = 360^\circ$ (1 p)

and therefore:

(12.6) $p = z_1 + z_2 - (\varphi_1 + \varphi_2)$ (1 p)

For the *NWA* triangle we can write that:

$$(12.7)\quad \frac{x_1}{d} = \frac{\sin(180^\circ - z_2 - \alpha)}{\sin p} \text{ and } \frac{x_2}{d} = \frac{\sin(180^\circ - z_1 - \alpha)}{\sin p} \qquad (2\,\text{p})$$

Thus:

$$(12.8)\quad x_1 = d\,\frac{\sin(z_2 + \alpha)}{\sin(z_1 + z_2 - (\varphi_1 + \varphi_2))} \text{ and } x_2 = d\,\frac{\sin(z_1 + \alpha)}{\sin(z_1 + z_2 - (\varphi_1 + \varphi_2))} \qquad (2\,\text{p})$$

D can be obtained either from the triangle *NOA* or *WOA* by the cosine theorem:

$$(12.9)\quad D^2 = R^2 + x_1^2 - 2Rx_1\cos(180^\circ - z_1)$$

or

$$(12.10)\quad D^2 = R^2 + x_2^2 - 2Rx_2\cos(180^\circ - z_2) \qquad (2\,\text{p})$$

Let us take the sum of them:

$$(12.11)\quad 2D^2 = 2R^2 + (x_1^2 + x_2^2) + 2R(x_1\cos z_1 + x_2\cos z_2), \qquad (1\,\text{p})$$

where we have used that $\cos(180^\circ - x) = -\cos x$.

(If someone determines D only from one of the two (*NOA*, *WOA*) triangles, 1 point can be deducted, because that means, we ignore the error bars of one of the measurements in a significant way.)

After substitution and dividing by two one gets:

$$(12.12)\quad D^2 = R^2 + d^2\,\frac{\sin^2(z_1 + \alpha) + \sin^2(z_2 + \alpha)}{2\sin^2(z_1 + z_2 - (\varphi_1 + \varphi_2))} + Rd\,\frac{\sin(z_2 + \alpha)\cos z_1 + \sin(z_1 + \alpha)\cos z_2}{\sin(z_1 + z_2 - (\varphi_1 + \varphi_2))} \qquad (3\,\text{p})$$

This contains only known quantities. After substitution we get (notice that for alpha and D we need the absolute values of the geographical latitudes):

$$\alpha = 55.5^\circ \qquad (1\,\text{p})$$

$$D = \boxed{65.8\,R} = \boxed{1.1\,d_{\text{Moon}}} \qquad (2\,\text{p})$$

So, the asteroid is at 65.8 Earth-radii from the centre of the Earth. This is slightly more than the distance to the Moon.

13. Distance to the Coma galaxy cluster

40 p

a) The mean radial velocity of the cluster, from the data given in the table, is

$$\langle v_\mathrm{r} \rangle \approx 6731\,\mathrm{km\,s^{-1}} \qquad (5\,\mathrm{p})$$

Using the Hubble law in the following form

$$v_\mathrm{r} = H_0 D, \qquad (2\,\mathrm{p})$$

where v_r is the mean radial velocity in km/s, D is the distance in Mpc, and $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ is the Hubble constant, we can get

$$D = \frac{v_\mathrm{r}}{H_0}$$

With numerical values:

$$D = \boxed{96\,\mathrm{Mpc}} \qquad (1\,\mathrm{p})$$

b) The angular diameter of the cluster is $\theta = 100' = 6000''$. Applying the definition of the parsec, $1''$ angular distance from 1 pc distance corresponds to 1 au (Astronomical Unit, $1\,\mathrm{au} \approx 1.5 \times 10^8\,\mathrm{km}$), i.e.

$$d\,[\mathrm{AU}] = \theta\,['']\times D\,[\mathrm{pc}] \qquad (3\,\mathrm{p})$$

Therefore, $\theta = 6000''$ from $D = 96\,\mathrm{Mpc} = 96 \times 10^6\,\mathrm{pc}$ gives (with $1\,\mathrm{Mpc} = 3.086 \times 10^{22}\,\mathrm{m}$):

$$d = 5.77 \times 10^{11}\,\mathrm{au} = 8.63 \times 10^{22}\,\mathrm{m} = \boxed{2.80\,\mathrm{Mpc}} \qquad (1\,\mathrm{p})$$

Alternative solution: $100' = 1.67^\circ = 0.0291\,\mathrm{rad}$. For such small angles $d \approx \theta D$. If θ is expressed in radians, then d and D have the same units. Thus, if $D = 96\,\mathrm{Mpc}$, then

$$d = 0.0291 \times 96\,\mathrm{Mpc} = \boxed{2.80\,\mathrm{Mpc}}$$

c) In the virial theorem

$$(13.1)\quad \langle K \rangle = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}m_i(v_i - v_m)^2$$

is the mean kinetic energy of N galaxies. The i^th galaxy has mass m_i and velocity v_i, and v_m is the mean velocity of the cluster.

$$(13.2)\quad \langle U \rangle = -\frac{3}{5}\frac{1}{N}\sum_{i=1}^{N}\frac{GM}{R}m_i$$

is the mean gravitational potential energy of N galaxies (having individual masses m_i as above) filling in a sphere of R radius, while G is the gravitational constant and $M = \sum_{i=1}^{N} m_i$ is the total mass of the cluster.

If the orbital velocity vectors are distributed randomly, then

$$(13.3)\quad \frac{1}{N}\sum_{i=1}^{N}(v_i - v_{\rm m})^2 = \frac{1}{N}\sum_{i=1}^{N}3(v_{i,\rm r} - v_{\rm m,r})^2 = \frac{N-1}{N}\sum_{i=1}^{N}\frac{3(v_{i,\rm r} - v_{\rm m,r})^2}{N-1} \to 3\sigma_{\rm r}^2,\ \text{if } N\to\infty,$$

where $v_{i,\rm r}$ is the radial velocity for the $i^{\rm th}$ galaxy, and $\sigma_{\rm r}$ is called the *velocity dispersion* of the cluster. Combining Eq. (13.1) and Eq. (13.3) one can get:

$$\langle K\rangle = \frac{3}{2}m\sigma_{\rm r}^2 \qquad (2\,\text{p})$$

Inserting $\langle K\rangle$ into the equation of virial theorem gives:

$$-2\times\frac{3}{2}m\sigma_{\rm r}^2 = -3m\sigma_{\rm r}^2 = \langle U\rangle \qquad (2\,\text{p})$$

Using Eq. (13.2) to express $\langle U\rangle$ one can get

$$\langle U\rangle = -\frac{3}{5}m\frac{GM}{R}\frac{1}{N}\sum_{i=1}^{N}1 = -\frac{3}{5}m\frac{GM}{R}, \qquad (2\,\text{p})$$

because $\sum_{i=1}^{N}1 = N$.

Inserting this expression into the one above, we get:

$$-3m\sigma_{\rm r}^2 = -\frac{3}{5}m\frac{GM}{R} \qquad (2\,\text{p})$$

Dividing by $-3m$ both sides, and expressing M we finally arrive at

$$M = \boxed{\frac{5R\sigma_{\rm r}^2}{G}}, \qquad (2\,\text{p})$$

which is the expression of the virial mass.

d) The radial velocity dispersion of the data given in the table is the same as the standard deviation of the data around the mean. The mean velocity, $\langle v_{\rm r}\rangle = 6731\,\text{km}\,\text{s}^{-1}$, was already derived in part a). Thus (depending on how we define the standard deviation):

$$\sigma_{{\rm r},N} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(v_{\rm r} - \langle v_{\rm r}\rangle\right)^2} = 1184\,\text{km}\,\text{s}^{-1}$$

or

$$\sigma_{{\rm r},N-1} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(v_{\rm r} - \langle v_{\rm r}\rangle\right)^2} = 1214\,\text{km}\,\text{s}^{-1} \qquad (10\,\text{p})$$

Inserting this value into the expression of the virial mass and using $R\approx 1.40\,\text{Mpc}$ as the cluster radius (from b)), the result is:

$$M_N = \frac{5\times(1.40\times 3.086\times 10^{22}\,\text{m})\times(1.184\times 10^{6}\,\text{m}\,\text{s}^{-1})^2}{6.67\times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}}$$

$$M_N = \boxed{4.53 \times 10^{45}\,\mathrm{kg}} \approx \boxed{2.28 \times 10^{15}\,\mathrm{M_\odot}}$$

or

$$M_{N-1} = \frac{5 \times (1.40 \times 3.086 \times 10^{22}\,\mathrm{m}) \times (1.214 \times 10^{6}\,\mathrm{m\,s^{-1}})^2}{6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}}$$

$$M_{N-1} = \boxed{4.77 \times 10^{45}\,\mathrm{kg}} \approx \boxed{2.40 \times 10^{15}\,\mathrm{M_\odot}}$$ (2 p)

e) Using the virial mass from d) and the given total luminosity of the cluster one can get:

$$\frac{M_N}{L} = \frac{2.28 \times 10^{15}\,\mathrm{M_\odot}}{5 \times 10^{12}\,\mathrm{L_\odot}} = 455\,\mathrm{M_\odot/L_\odot} \approx 460\,\mathrm{M_\odot/L_\odot}$$

or

$$\frac{M_{N-1}}{L} = \frac{2.40 \times 10^{15}\,\mathrm{M_\odot}}{5 \times 10^{12}\,\mathrm{L_\odot}} = 479\,\mathrm{M_\odot/L_\odot} \approx 480\,\mathrm{M_\odot/L_\odot}$$ (1 p)

This mass-luminosity ratio is much higher than that of the individual galaxies which is typically in between 1 and 100. (1 p)

f) (A) and (D) (4 p)

14. Photographing a nanosatellite

60 p

a) As it was suggested in the hint the key is a comparison of the sunlight scattered from a 10 cm × 10 cm aluminium plate and the Moon to the observer (both are nearly at the local zenith).

Let the distance of the Moon and the satellite from the observing site be d_{Moon}, and d_{MASAT}, respectively. Under the mentioned "ideal" circumstances, when both the Moon and the satellite is near the local zenith: $d_{\mathrm{MASAT}} = 900\,\mathrm{km}$, while

(14.1) $d_{\mathrm{Moon}} = s_{\mathrm{Moon}} - R_{\mathrm{Earth}} - R_{\mathrm{Moon}},$ (2 p)

where the average distance between the centre of Earth and Moon s_{Moon} is included in the *Table of constants.*

Let the intensity coming from the Sun and scattered back to the direction of the observer be I, this is considered practically the same at the solar distance of the Moon and the satellite, but necessarily multiplied by the albedo of the given surface quality (the lunar albedo a_{Moon} is given in the Constant table, while a_{MASAT} of the satellite is in the text).

Finally, for using the magnitude formula, to find the observable magnitude of the satellite, we have to consider the resulted flux of the scattered light at the observer, which is directly proportional to the intensity, the albedo and the scattering area, but is inversely proportional to the distance on the square:

(14.2) $I_{\mathrm{scat}} \approx \dfrac{a_{\mathrm{obj}} I A_{\mathrm{obj}}}{d_{\mathrm{obj}}^2}$ (3 p)

Thus, the magnitude formula gives:

(14.3) $m_{\mathrm{MASAT}} - m_{\mathrm{Moon}} = -2.5 \log \dfrac{I_{\mathrm{MASAT}}}{I_{\mathrm{Moon}}} = -2.5 \log \dfrac{\dfrac{a_{\mathrm{MASAT}} I A_{\mathrm{MASAT}}}{d_{\mathrm{MASAT}}^2}}{\dfrac{a_{\mathrm{Moon}} I A_{\mathrm{Moon}}}{d_{\mathrm{Moon}}^2}}$ (3 p)

With numerical values:

$m_{\mathrm{MASAT}} = \boxed{9.72^{\mathrm{m}}}$ (2 p)

b) A moving point-like object will generally cause a line on the CCD image, resulting less photoelectrons in a given pixel, than a similarly bright unmoving object. Since the MASAT–1 satellite is so small, it is not resolved by the telescope – it will cause a seeing-sized spot at the focus of a telescope, like any other stars on the sky, but since the text of the problem described: we should omit the seeing. Thus, the image of any point source will be formed by the diffraction – so we shall use the approximate formula of the Airy disk.

By the diffraction-limited imaging, the point-like sources illuminate a spot in the focal plane having the angular diameter of the so-called Airy disc determined by the formula:

(14.4) $\varphi_{\mathrm{Airy}} = 2.44\,\dfrac{\lambda}{D},$ (2 p)

while its linear size at the focus of f focal length telescope is:

(14.5) $d_{\mathrm{Airy}} = f\varphi_{\mathrm{Airy}}$ (2 p)

For the telescope of Baja Observatory, described in the text, at a typical visual wavelength of 0.55 µm:

$$d_{\text{Airy}} = 2.44\,\frac{f\lambda}{D} = 2.44 \times \frac{4200\,\text{mm} \times 0.55\,\mu\text{m}}{500\,\text{mm}} = 11.27\,\mu\text{m} \qquad (2\,\text{p})$$

This is very close to the pixel size of the applied CCD. However, this number is not necessary for our calculation, since both the stars and the MASAT–1 are point-like sources, illuminating the same number of pixels, thus the number of pixels will drop out from all comparative calculations. So does the quantum efficiency value of the applied CCD.

For deriving the effect of the moving object, let us first calculate the scale of the image at the focus of the 50 cm RC telescope of Baja (N is the f-number of the telescope, $N = f/D$):

$$S = \frac{206\,265''}{DN} = \frac{206\,265''}{500\,\text{mm} \times 8.4} = 49\,''\,\text{mm}^{-1} = 0.049\,''\,\mu\text{m}^{-1} \qquad (3\,\text{p})$$

This means, that one pixel of the attached CCD describes $P = 0.049\,''\,\mu\text{m}^{-1} \times 9\,\mu\text{m} = 0.44''$ side length area on the sky. (3 p)

Now let us estimate the relative flux, resulted by a star having the limiting magnitude of the telescope-CCD system (19.5^{m}), by the magnitude formula:

(14.6) $19.5 = -2.5 \log \Phi_{19.5} \rightarrow \Phi_{19.5} = 10^{-7.8}\,\text{s}^{-1}$ (2 p)

And similarly, for the MASAT–1 brightness – observed with the same system, under the same conditions, if it were an unmoving object:

(14.7) $9.72 = -2.5 \log \Phi_{\text{MASAT}} \rightarrow \Phi_{\text{MASAT}} = 10^{-3.9}\,\text{s}^{-1}$ (2 p)

Next, we have to calculate the angular speed of MASAT–1 at 900 km altitude above the observing site, in order to derive the fluxes falling onto one pixel of the CCD on the telescope.

The velocity of the motion on a circular orbit with orbital radius r:

(14.8) $v_r = \sqrt{\frac{GM_{\text{Earth}}}{r}}$, (2 p)

where $r = h + R_{\text{Earth}}$. (2 p)

With numerical values:

$$v_r = 7404\,\text{m}\,\text{s}^{-1} \qquad (1\,\text{p})$$

Hence, the visible angular speeds (aside from the Earth rotation):

(14.9) $\omega_r = \frac{v_r}{r}$ (2 p)

With numerical values:

$$\omega_r = 0.47\,\text{deg s}^{-1} = 1696.8\,''\,\text{s}^{-1}, \qquad (1\,\text{p})$$

when the satellite is near the local zenith.

For deriving the relative speed, which will be really exposed onto the CCD, we have to calculate the angular speed of the observing site caused by the terrestrial rotation. At the latitude of φ it is given as:

(14.10) $\omega_{\text{rot},\varphi} = \omega_{\text{rot,eq}} \cos\varphi$ (2 p)

For Baja its numerical value:

$$\omega_{\rm rot,B} = 10.4''\,{\rm s}^{-1} \qquad (1\,{\rm p})$$

Its component parallel to the direction of the satellite's orbital motion:

$$(14.11)\quad \omega_{\rm rot,B,\it i} = \omega_{\rm rot,B}\cos i_{\rm MASAT} \qquad (2\,{\rm p})$$

For Baja its numerical value:

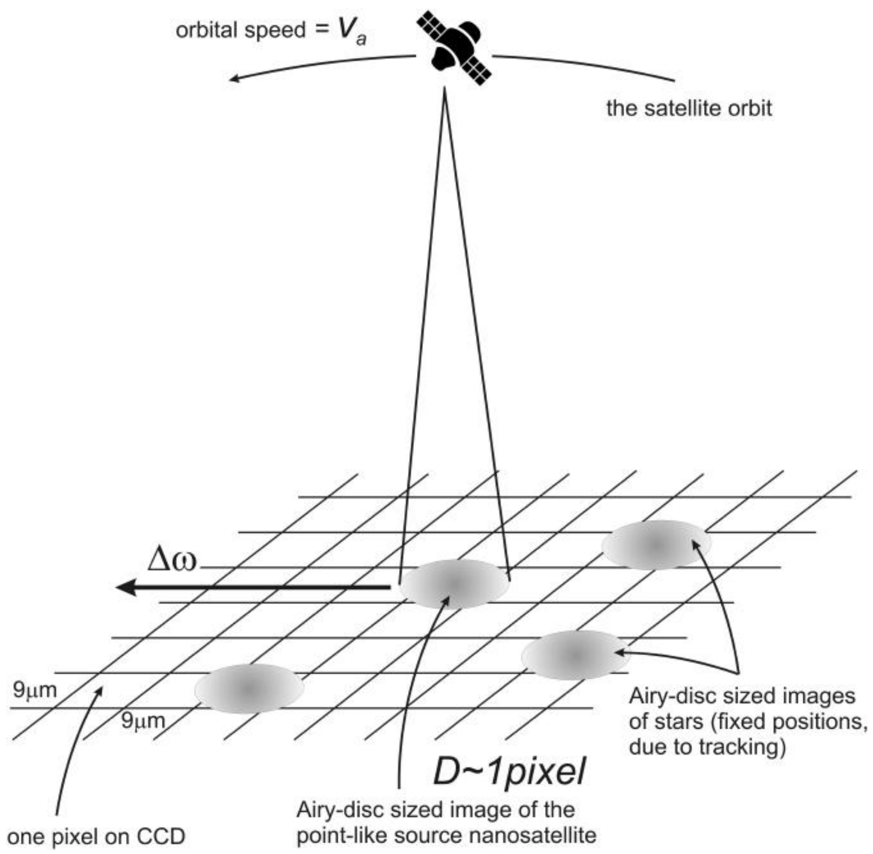
$$\omega_{\rm rot,B,\it i} = 3.6''\,{\rm s}^{-1} \qquad (1\,{\rm p})$$

Finally, the relative angular speed of the nanosatellite above the telescope:

$$(14.12)\quad \Delta\omega = \omega_a - \omega_{\rm rot,B,\it i} \qquad (2\,{\rm p})$$

With numerical values:

$$\Delta\omega = 1693.3''\,{\rm s}^{-1} \qquad (1\,{\rm p})$$



Since the satellite image is moving with a given $\Delta\omega$ angular speed, this means that the flux coming from the satellite is falling onto a given pixel only for a very limited time (independently of the exposure time of the image, but acting similarly, than that). Considering our very simple approximation we get an approximate "illumination time" for one pixel falling onto the line of the orbit:

$$(14.13)\quad \tau = \frac{P}{\Delta\omega} \qquad (2\,{\rm p})$$

With numerical values:

$$\tau = \frac{0.44''}{1693.3''\,{\rm s}^{-1}} = 2.61\times10^{-4}\,{\rm s} \qquad (1\,{\rm p})$$

For finishing our estimation – to get the magnitude level of any pixel on the observable trace of this nanosatellite, let us use the magnitude formula for them and one non-moving light source at the detection limit. For doing this we have to multiply the relative fluxes Φ_{MASAT} and $\Phi_{19.5}$ with the respective illumination times. (For a 19.5^{m} stars we should use the 2 min-long exposure time mentioned in the text, which is needed to collect enough photoelectrons for the safe detection of such a faint star.)

(14.14) $$m_{\mathrm{MASAT,trace}} - 19.5^{\mathrm{m}} = -2.5 \log \frac{\tau \Phi_{\mathrm{MASAT}}}{\tau_{\exp} \Phi_{19.5}}$$ (2 p)

With numerical values:

$$m_{\mathrm{MASAT,trace}} = \boxed{23.9^{\mathrm{m}}}$$ (2 p)

So, the trace of the moving 23.9^{m} magnitude nanosatellite looks so faint on a (2 min-long exposure) CCD image, as a $\sim 24^{\mathrm{m}}$ stars would be seen, but they fall surely below the detection limit for the described telescope. Thus, the answer is definitely

NO, WE COULD NOT RECORD THE MASAT–1 (2 p)

using that telescope.

c) If we consider a real atmosphere, due to the effect of the atmospheric turbulences (the so-called seeing) during the 2 minutes-long exposure the images of the stars will never be described by the ideal Airy disk, but expanded to a much larger "seeing spot", which can be approximated by a Gaussian profile. As mentioned in the text: in Hungary, its diameter at half-maxima is typically $3.5''$. This means, that the flux coming from a 19.5^{m} star will be distributed over about a

$$\frac{\mathrm{FWHM}}{P} \times \frac{\mathrm{FWHM}}{P} = \frac{3.5''}{0.44''} \times \frac{3.5''}{0.44''} \approx 8 \times 8$$ (1 p)

pixels area. Thus, now the $F_{19.5}$ flux will be distributed amongst $4 \cdot 4\pi \approx 50$ pixels (for the sake of simplicity: homogeneously), i.e. one pixel gets:

(14.15) $$\Phi_{19.5} = \frac{10^{-7.8}}{50}\,\mathrm{s}^{-1} \approx 10^{-9.5}\,\mathrm{s}^{-1}$$ (1 p)

As we have seen above, the satellite image is moving so fast along the focal plane (0.26 ms per a pixel), that during this short time, the atmosphere can be considered to be still. Thus, the image of MASAT–1 can further be approximated with the Airy-disc size also in a real atmosphere.

Let us correct our last magnitude formula for this fact:

(14.16) $$m_{\mathrm{MASAT,trace}} - 19.5^{\mathrm{m}} = -2.5 \log \frac{\tau \Phi_{\mathrm{MASAT}}}{\tau_{\exp} \dfrac{\Phi_{19.5}}{50}}$$ (2 p)

With numerical values:

$$m_{\mathrm{MASAT,trace}} = \boxed{19.6^{\mathrm{m}}}$$ (2 p)

Under real conditions the situation is much better: the trace of the small nanosatellite is near the detectability! Thus, owing to further secondary effects (pixel sensitivity inhomogenities, temporarily changing transparency of the atmosphere, etc.), the trace of the cubesat can easily be seen on the CCD of the 50 cm reflector. So in real cases, the answer can be YES.

YES, WE COULD HAVE RECORDED THE MASAT–1. (2 p)